Euler's Formula for Fractional Powers of i

Jerrold Prothero, Ph.D.
jdprothero@gmail.com

April 15, 2007

To Leonhard Euler, on the occasion of his 300^{th} birthday.

Having lived, they live forever,
Who to purpose grand endeavor.
And nor shall time, nor tongue of man,
E'er taint or burnish that rare clan;
For none may mark them with their sign,
Who chose to breathe the breath divine.

Contents

1 Introduction **2**

2 The Magic Suitcase **4**

3 The Geometric Interpretation **6**

4 The Taylor Series Interpretation **9**

5 The Algebraic Interpretation **12**

6 Projection Onto Cairns Space **18**

7 The Broom Theorems **21**
- 7.1 Sweeping Up 21
- 7.2 Sweeping Down 22
- 7.3 Sweeping Sideways 23

8 Sundry Findings **25**
- 8.1 Pythagorean Identities 25
- 8.2 Curvature 27
- 8.3 The Inner Product on Level m 28
- 8.4 Differentiation, Rotation and Displacement 29
- 8.5 The Roots of n^{th}-Degree Polynomials 31

9 Conclusion **34**

A Derivation of $\psi_{mn}(x)$ **35**

B Proof of the Equivalence of $\psi_{mn}(x)$ and $E_{mn}(x)$ **37**

C Factoring $E_{mn}(x)$ **41**
- C.1 Derivation of $E_{m,0}$ Factors 41
- C.2 $E_{mn}(x)$ 47

D The Orthogonality of $\psi_{mn}(x)$ Coefficients **48**

E Derivation of the Broom Theorems **51**
E.1 Sweeping Up . . . 51
E.2 Sweeping Sideways . . . 53

F Derivation of the Inner Product **56**

G The Cairns Functions For $m \leq 4$ **62**
G.1 $m \leq 0$ . . . 62
G.2 $m = 1$ . . . 63
G.3 $m = 2$ . . . 63
G.4 $m = 3$ . . . 64
G.5 $m = 4$ . . . 65

Abstract

A generalization of Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ for fractional powers of i is discussed. This generalization, based on the term

$$e^{xi^{\left(2^{2-m}\right)}}$$

corresponds to a sequence of spirals in the complex plane. Just as e^{ix} generates $\cos(x)$ and $\sin(x)$, $e^{xi^{\left(2^{2-m}\right)}}$ generates sets of functions ("levels") which generalize the properties of $\cos(x)$ and $\sin(x)$ by combining periodic and exponential characteristics. The set of all functions generated by $e^{xi^{\left(2^{2-m}\right)}}$ defines an orthogonal basis for the set of all Taylor polynomials.

Chapter 1

Introduction

Asclepius, in days when we are young,
The Music of the Spheres we first hear sung!
Through straining then, to listen and to learn,
What revelries of truth one may discern!

According to Richard Feynman, Euler's formula[1]

$$e^{ix} = \cos(x) + i\sin(x) \tag{1.1}$$

is the most remarkable in mathematics.

Certainly, it is one of the most important, relating the exponential constant e, which describes growth, to the $\cos(x)$ and $\sin(x)$ functions, which describe periodic behavior, through the constant i, which describes rotations.

It is odd, given the importance of the equation, that there does not appear to exist a systematic discussion of the implications of raising i in Euler's formula to fractional powers. As will be shown here, there is a natural extension of Euler's formula for fractional powers of i that provides coherent and interesting results.

Implications include

1. While Euler's formula corresponds to a circle in the complex plane, fractional powers of i correspond to spirals.

2. By varying a single parameter, it is possible to generate e^x, e^{-x}, $\cos(x)$, $\sin(x)$, and an infinite family of related functions.

3. Just as e^{ix} generates $\cos(x)$ and $\sin(x)$, each fractional power of i considered here generates its own set of functions (a "level"). The levels have related but distinct properties.

4. The set of functions at each level form a differential cycle. That is, every function in the cycle is the derivative of the next higher function in the cycle, with the highest function being the negative derivative of the lowest.

[1] http://en.wikipedia.org/wiki/Euler%27s_formula

5. Each generated function can be represented in terms of a combination of hyperbolic and circular components. More precisely, every such function is either e^x, e^{-x}, $\cos(x)$, $\sin(x)$, or a sum of products of either $\cosh(x)$ or $\sinh(x)$ with either $\cos(x)$ or $\sin(x)$.

6. The set of functions generated at all levels defines an orthogonal basis for the Taylor polynomials.

7. A geometric curvature can be associated with Taylor polynomials.

8. In certain cases, the roots of n^{th}–degree polynomials can be found very simply.

I have attempted to make this monograph accessible to as wide an audience as possible: mathematics should not be left solely to specialists. References to background topics are collected online at *http://del.icio.us/jprothero/Euler*

See also *jerroldprothero.blogspot.com*

Chapter 2

The Magic Suitcase

Of all fractions, perhaps the simplest are those which are powers of two ($\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc.). Consequently, in considering possible generalizations of Euler's term e^{ix} to fractional powers of i we are led naturally to the term

$$\boxed{e^{xi^{\left(2^{2-m}\right)}}} \tag{2.1}$$

Where m is an integer. I call this the "magic suitcase", because however much one unpacks from this term, there is always more to be found.

While perhaps initially mysterious, the suitcase simply takes advantage of the useful properties of powers of two, which interact very naturally with i. Each value of m corresponds to a particular level, whose characteristics are outlined below.

- *For* $m = 0$ (and, similarly, for any negative value of m) the i exponent reduces to

  $$i^{(2^{2-0})} = i^{(2^2)} = i^4 = 1$$

  so the suitcase is equal to the familiar natural exponential, e^x.

- *For* $m = 1$, the suitcase reduces to

  $$e^{xi^{\left(2^{2-1}\right)}} = e^{xi^{\left(2^{1}\right)}} = e^{xi^2} = e^{-x}$$

- *For* $m = 2$, we have

  $$e^{xi^{\left(2^{2-2}\right)}} = e^{xi^{\left(2^{0}\right)}} = e^{xi^1} = e^{ix}$$

This gives us the traditional Euler term, corresponding to a circle in the complex plane.

- *For* $m = 3$*,* we have

$$e^{xi^{\left(2^{2-3}\right)}} = e^{xi^{\left(2^{-1}\right)}} = e^{xi^{1/2}}$$

which (as we will see) corresponds to a spiral in the complex plane.

- *For* $m = 4$*,* we have $e^{xi^{1/4}}$, corresponding to a (different) spiral in the complex plane.

- And so on. As m becomes very large, we have

$$\lim_{m\to\infty} i^{\left(2^{2-m}\right)} = i^0 = 1$$

so

$$\lim_{m\to\infty} e^{xi^{\left(2^{2-m}\right)}} = e^x$$

The suitcase term both "starts" ($m = 0$) and "ends" ($m \to \infty$) with e^x. We will soon see a geometric interpretation of why this is true.

The line of thought discussed here was originally inspired by the term

$$e^{\theta i^{(2^m - 1)/2^{m-1}}} \tag{2.2}$$

provided by John Cairns in U.S. Patent 5,563,556, *Geometrically Modulated Waves.*[1] Aside from a certain ungainliness, the Cairns term has technical limitations and the patent itself contains several algebraic errors.

Nonetheless, so far as I know (and, apparently, so far as the Patent Office knew), Cairns was the first to examine the consequences of fractional powers of i in Euler's formula. Consequently, it is appropriate to refer to the set of functions described below as "Cairns space".

[1] http://www.google.com/patents?vid=USPAT5563556

Chapter 3

The Geometric Interpretation

A well-known consequence of Euler's formula is that $e^{i\pi/2} = i$.[1] Therefore

$$i^{(2^{2-m})} = (e^{i\pi/2})^{(2^{2-m})} \tag{3.1}$$

By replacing $i^{(2^{2-m})}$ with $(e^{i\pi/2})^{(2^{2-m})}$ in the suitcase, we can express it equivalently as

$$\begin{aligned}
e^{xi^{\left(2^{2-m}\right)}} &= e^{x\left(e^{i\pi/2}\right)^{\left(2^{2-m}\right)}} && (3.2)\\
&= e^{xe^{(i\pi/2)\left(2^{2-m}\right)}} && (3.3)\\
&= e^{xe^{i\pi\left(2^{1-m}\right)}} && (3.4)
\end{aligned}$$

We can now invoke Euler's formula to break apart the upper exponent:

$$e^{i\pi\left(2^{1-m}\right)} = \cos(\pi 2^{1-m}) + i\sin(\pi 2^{1-m}) \tag{3.5}$$

By plugging Equation (3.5) in for the exponent of Equation (3.4), we get

[1]This follows from inserting $\pi/2$ into Euler's formula: $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$, so $e^{i\pi/2} = \cos(\pi/2) + i\sin(\pi/2) = i$.

$$e^{xe^{i\pi\left(2^{1-m}\right)}} = e^{x(\cos(\pi 2^{1-m})+i\sin(\pi 2^{1-m}))} \tag{3.6}$$
$$= e^{x\cos(\pi 2^{1-m})} e^{ix\sin(\pi 2^{1-m})} \tag{3.7}$$

and therefore

$$\boxed{e^{xi^{\left(2^{2-m}\right)}} = e^{x\cos(\pi 2^{1-m})} e^{ix\sin(\pi 2^{1-m})}} \tag{3.8}$$

The first factor of Equation (3.8), $e^{x\cos(\pi 2^{1-m})}$, is a pure real-valued exponential. The second factor, $e^{ix\sin(\pi 2^{1-m})}$, defines a circle in the complex plane with period $2\pi/\sin(\pi 2^{1-m})$. Multiplying these two factors together, we have a "growing circle": that is, a spiral.

As m increases, we have

$$\lim_{m\to\infty} \cos(\pi 2^{1-m}) = 1 \tag{3.9}$$
$$\lim_{m\to\infty} \sin(\pi 2^{1-m}) = 0 \tag{3.10}$$

Consequently, with increasing m the exponential term grows faster, and the spiral rotation "slows down" (period increases).

Regardless of the value of m, for $x = 0$ we have $e^{xi^{\left(2^{2-m}\right)}} = e^0 = 1$. Therefore, every spiral "starts" on the real axis of the complex plane, at the point $(1, 0)$. As m increases, the period also increases: in the limit of large m, the spiral "slows down" its rotation to the degree that it never gets off the real axis. This is the geometric interpretation of the fact that $e^{xi^{\left(2^{2-m}\right)}}$ converges back to e^x as $m \to \infty$.

For clarity, let us look at Equation (3.8) for particular values of m.

- *For* $m = 0$ (and similarly for any negative value of m) $\cos(\pi 2^{1-m}) = 1$ and $\sin(\pi 2^{1-m}) = 0$, so
$$e^{x\cos(\pi 2^{1-m})} e^{ix\sin(\pi 2^{1-m})} = e^x$$

- *For* $m = 1$, $\cos(\pi 2^{1-m}) = -1$ and $\sin(\pi 2^{1-m}) = 0$, so
$$e^{x\cos(\pi 2^{1-m})} e^{ix\sin(\pi 2^{1-m})} = e^{-x}$$

- *For $m = 2$,* $\cos(\pi 2^{1-m}) = 0$ and $\sin(\pi 2^{1-m}) = 1$, so

$$e^{x\cos(\pi 2^{1-m})}e^{ix\sin(\pi 2^{1-m})} = e^{ix}$$

- *For large m,* $\lim_{m\to\infty} \cos(\pi 2^{1-m}) = 1$ and $\lim_{m\to\infty} \sin(\pi 2^{1-m}) = 0$, so

$$e^{x\cos(\pi 2^{1-m})}e^{ix\sin(\pi 2^{1-m})} = e^{x}$$

These results are of course consistent with those found in Chapter 1.

Incidentally, by invoking the Euler formula and its equivalent for hyperbolic trigonometry

$$e^x = \cosh(x) + \sinh(x) \tag{3.11}$$

we can decompose Equation (3.8) into

$$\begin{aligned} e^{xi^{\left(2^{2-m}\right)}} &= (\cosh(\cos(\pi 2^{1-m})x) + \sinh(\cos(\pi 2^{1-m})x)) \cdot \\ &\quad (\cos(\sin(\pi 2^{1-m})x) + i\sin(\sin(\pi 2^{1-m})x)) \end{aligned} \tag{3.12}$$

As we shall see, this balanced treatment of hyperbolic and circular functions is intrinsic to Cairns space.

Chapter 4

The Taylor Series Interpretation

It is well-known that Euler's formula can be derived by expanding e^{ix} as a Taylor series and grouping terms. One gets

$$\begin{aligned}
e^{ix} &= \sum_{t=0}^{\infty} \frac{(ix)^t}{t!} && (4.1)\\
&= 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \ldots && (4.2)\\
&= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \ldots && (4.3)\\
&= (1 - \frac{x^2}{2!} + \ldots) + i(x - \frac{x^3}{3!} + \ldots) && (4.4)\\
&= \sum_{t=0}^{\infty} \frac{x^{2t}}{(2t)!} + i \sum_{t=0}^{\infty} \frac{x^{2t+1}}{(2t+1)!} && (4.5)\\
&= \cos(x) + i\sin(x) && (4.6)
\end{aligned}$$

Precisely the same procedure can be followed to generate functions for any m-value of the suitcase. For $m = 3$, we have

$$
\begin{aligned}
e^{xi^{1/2}} &= \sum_{t=0}^{\infty} \frac{(xi^{1/2})^t}{t!} && (4.7)\\
&= 1 + (xi^{1/2}) + \frac{(xi^{1/2})^2}{2!} + \frac{(xi^{1/2})^3}{3!} + && (4.8)\\
&\quad \frac{(xi^{1/2})^4}{4!} + \frac{(xi^{1/2})^5}{5!} + \frac{(xi^{1/2})^6}{6!} + \frac{(xi^{1/2})^7}{7!} + \dots && (4.9)\\
&= 1 + i^{1/2}x + i^1\frac{x^2}{2!} + i^{3/2}\frac{x^3}{3!} + && (4.10)\\
&\quad - \frac{x^4}{4!} - i^{1/2}\frac{x^5}{5!} - i^1\frac{x^6}{6!} - i^{3/2}\frac{x^7}{7!} + \dots && (4.11)\\
&= (1 + -\frac{x^4}{4!} + \dots) + i^{1/2}(x - \frac{x^5}{5!} + \dots) + && (4.12)\\
&\quad i(\frac{x^2}{2!} - \frac{x^6}{6!} + \dots) + i^{3/2}(\frac{x^3}{3!} - \frac{x^7}{7!} + \dots) && (4.13)\\
&= \psi_{3,0}(x) + i^{1/2}\psi_{3,1}(x) + i\psi_{3,2}(x) + i^{3/2}\psi_{3,3}(x) && (4.14)
\end{aligned}
$$

Where

$$
\begin{aligned}
\psi_{3,0}(x) &\equiv \sum_{t=0}^{\infty} \frac{x^{4t}}{(4t)!} && (4.15)\\
\psi_{3,1}(x) &\equiv \sum_{t=0}^{\infty} \frac{x^{4t+1}}{(4t+1)!} && (4.16)\\
\psi_{3,2}(x) &\equiv \sum_{t=0}^{\infty} \frac{x^{4t+2}}{(4t+2)!} && (4.17)\\
\psi_{3,3}(x) &\equiv \sum_{t=0}^{\infty} \frac{x^{4t+3}}{(4t+3)!} && (4.18)\\
& && (4.19)
\end{aligned}
$$

The general relationship for any m is[1]

$$
\boxed{e^{xi^{\left(2^{2-m}\right)}} = \sum_{n=0}^{\lceil 2^{m-1} \rceil - 1} i^{n2^{2-m}} \psi_{mn}(x)} \qquad (4.20)
$$

[1]The proof of this equation is simply a generalization of the above $m = 3$ discussion: see Appendix A.

where

$$\boxed{\psi_{mn}(x) = \sum_{t=0}^{\infty} (-1)^{t\lceil 2^{1-m}\rceil} \frac{x^{t\lceil 2^{m-1}\rceil + n}}{(t\lceil 2^{m-1}\rceil + n)!}} \tag{4.21}$$

The ceiling functions are there to handle boundary cases correctly. The ceiling function prevents 2^{m-1} from taking on a fractional value for $m \leq 0$. In Equation (4.21) the factor $\lceil 2^{1-m}\rceil$ "turns off" the sign alternation of terms for $m \leq 0$, corresponding the function e^x.

Here are some interesting points about Equations (4.20) and (4.21).

- The value of m determines a "level" of Cairns space, defining a set of functions.
  - $m < 0$ corresponds to the single function $\psi_{0,0}(x) = e^x$
  - $m = 1$ corresponds to the single function $\psi_{1,0}(x) = e^{-x}$
  - $m = 2$ corresponds to the two functions $\psi_{2,0}(x) = \cos(x)$ and $\psi_{2,1}(x) = \sin(x)$
  - $m = 3$ corresponds to the four functions $\psi_{3,0}(x)$, $\psi_{3,1}(x)$, $\psi_{3,2}(x)$, and $\psi_{3,3}(x)$, as defined above
  - $m = 4$ corresponds to eight functions
  - In general, for any m there are $\lceil 2^{m-1}\rceil$ functions.
- The value of n determines a particular function at level m. The possible values of n run from 0 to $\lceil 2^{m-1}\rceil - 1$.
- At level m, the Taylor series corresponding to $\psi_{mn}(x)$ consist of terms with powers differing by $\lceil 2^{m-1}\rceil$ and offset by n.
- If n is even, all terms in $\psi_{mn}(x)$ have even power; if n is odd, all terms in $\psi_{mn}(x)$ have odd power. Consequently, $\psi_{mn}(x)$ is symmetric for even n and anti-symmetric for odd n.[2]
- For $m > 1$, we see by inspection that the functions in level m form a *differential cycle*. That is, for $n > 0$ $\psi'_{m,n}(x) = \psi_{m,n-1}(x)$ and for $n = 0$ we have $\psi'_{m,0}(x) = -\psi_{m,\lceil 2^{m-1}\rceil - 1}(x)$

[2]That is, for even n $\psi_{mn}(-x) = \psi_{mn}(x)$ and for odd n $\psi_{mn}(-x) = -\psi_{mn}(x)$. This property is familiar from the well-known behavior of $\psi_{2,0}(x) = \cos(x)$ and $\psi_{2,1}(x) = \sin(x)$.

Chapter 5

The Algebraic Interpretation

In Chapter 4, we saw how to express the functions generated by $e^{xi^{\left(2^{2-m}\right)}}$ as Taylor series. Is it also possible to describe these same functions in terms of familiar algebraic expressions? In this chapter, we shall see that it is.

From Chapter 4 we know that

$$\psi_{0,0}(x) = e^{x} = e^{xi^{4}} \tag{5.1}$$

$$\psi_{1,0}(x) = e^{-x} = e^{xi^{2}} \tag{5.2}$$

$$\psi_{2,0}(x) = cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \frac{1}{2}\left(e^{xi^{1}} + e^{xi^{3}}\right) \tag{5.3}$$

Note that

$$2^{2-m} = \begin{cases} 4 & \text{if } m = 0; \\ 2 & \text{if } m = 1; \\ 1 & \text{if } m = 2. \end{cases} \tag{5.4}$$

So the above equations all have $e^{xi^{\left(2^{2-m}\right)}}$ as the first term. Notice also that the above three equations can be described by the rules

- Start with the term $e^{xi^{\left(2^{2-m}\right)}}$

- If $\left(2^{m-2}\right)(1+2) < 4$, add the term $e^{xi^{(2^{m-2})(1+2)}}$
- Divide by the number of terms

This might lead one to guess[1] that

$$\psi_{3,0}(x) \stackrel{?}{=} \frac{1}{4}\left(e^{xi^{1/2}} + e^{xi^{3/2}} + e^{xi^{5/2}} + e^{xi^{7/2}}\right) \tag{5.5}$$

$$\psi_{4,0}(x) \stackrel{?}{=} \frac{1}{8}(\quad e^{xi^{1/4}} + e^{xi^{3/4}} + e^{xi^{5/4}} + e^{xi^{7/4}} + \tag{5.6}$$

$$e^{xi^{9/4}} + e^{xi^{11/4}} + e^{xi^{13/4}} + e^{xi^{15/4}}) \tag{5.7}$$

and that in general[2]

$$\psi_{m,0}(x) \stackrel{?}{=} \frac{1}{\lceil 2^{m-1} \rceil} \sum_{p=0}^{\lceil 2^{m-1} \rceil - 1} e^{xi^{(2p+1)2^{2-m}}} \tag{5.8}$$

To prove Equation (5.8), it is useful to *define* a function $E_{m,0}(x)$ with the above property. By definition

$$E_{m,0}(x) \equiv \frac{1}{\lceil 2^{m-1} \rceil} \sum_{p=0}^{\lceil 2^{m-1} \rceil - 1} e^{xi^{(2p+1)2^{2-m}}} \tag{5.9}$$

If we assume $E_{m,0}(x) = \psi_{m,0}(x)$, it follows that the derivative and integrals of $E_{m,0}(x)$ must be equal to the derivatives and integrals of $\psi_{m,0}(x)$. By inspection, taking the integrals of Equation (5.9) with integration constant zero, we define

[1] "hypothesize"

[2] There is a tension between simple exposition, on the one hand, and a certain realism about how results are actually found on the other. In practice, I found the $m = 3$ relationship by means less than pretty, not suitable for a family monograph, then guessed the general relationship.

$$E_{mn}(x) \equiv \frac{1}{\lceil 2^{m-1} \rceil} \sum_{p=0}^{\lceil 2^{m-1} \rceil - 1} i^{-n(2p+1)2^{2-m}} e^{xi^{(2p+1)2^{2-m}}} \tag{5.10}$$

With the above definition of $E_{mn}(x)$, we can then prove that for all m,

$$\psi_{mn}(x) = E_{mn}(x) \tag{5.11}$$

as is shown in Appendix B.[3]

The equivalence of $\psi_{mn}(x)$ and $E_{mn}(x)$ is quite interesting, as it relates an infinite set of Taylor series to a corresponding infinite set of exponential functions. One has two different ways of looking at the component functions of $e^{xi^{\left(2^{2-m}\right)}}$.

Because $E_{mn}(x) = \psi_{mn}(x)$, by comparison with Equation (4.20) we must have

$$e^{xi^{\left(2^{2-m}\right)}} = \sum_{n=0}^{\lceil 2^{m-1} \rceil - 1} i^{n2^{2-m}} E_{mn}(x) \tag{5.12}$$

Since $\psi_{mn}(x)$ is always a real-valued function, it follows that $E_{mn}(x)$ must also be always real-valued. This is rather surprising, given that for $n \neq 0$ the terms of $E_{mn}(x)$ have imaginary coefficients. With each successive differentiation of $E_{mn}(x)$ a shower of imaginary factors descend from the exponents to become coefficients; and yet the imaginary part of $E_{mn}(x)$ always exactly cancels out to produce a purely real result. It is a most delicate dance.

Let us examine $E_{mn}(x)$ in more detail. For $m = 3$, we have

$$E_{3,0}(x) = \frac{1}{4}\left(e^{xi^{1/2}} + e^{xi^{3/2}} + e^{xi^{5/2}} + e^{xi^{7/2}}\right) \tag{5.13}$$

We can use the method of Chapter 3 to write

[3]Essentially, the proof amounts to expanding $E_{m,0}$ as a sum of Taylor series, then recursively cancelling terms until one is left with $\psi_{m,0}$. The equivalence for all n follows from parallel differentiation.

$$
\begin{aligned}
i^{1/2} &= \left(e^{i\pi/2}\right)^{1/2} && (5.14)\\
&= e^{i\pi/4} && (5.15)\\
&= \cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}) && (5.16)\\
&= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} && (5.17)
\end{aligned}
$$

$$
\begin{aligned}
i^{3/2} &= \left(e^{i\pi/2}\right)^{3/2} && (5.18)\\
&= e^{i3\pi/4} && (5.19)\\
&= \cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4}) && (5.20)\\
&= -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} && (5.21)
\end{aligned}
$$

$$
\begin{aligned}
i^{5/2} &= \left(e^{i\pi/2}\right)^{5/2} && (5.22)\\
&= e^{i5\pi/4} && (5.23)\\
&= \cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4}) && (5.24)\\
&= -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} && (5.25)
\end{aligned}
$$

$$
\begin{aligned}
i^{7/2} &= \left(e^{i\pi/2}\right)^{7/2} && (5.26)\\
&= e^{i7\pi/4} && (5.27)\\
&= \cos(\frac{7\pi}{4}) + i\sin(\frac{7\pi}{4}) && (5.28)\\
&= \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}} && (5.29)
\end{aligned}
$$

And hence

$$E_{3,0}(x) = \frac{1}{4}(e^{x/\sqrt{2}}e^{xi/\sqrt{2}} + e^{-x/\sqrt{2}}e^{xi/\sqrt{2}} + e^{-x/\sqrt{2}}e^{-xi/\sqrt{2}} + e^{x/\sqrt{2}}e^{-xi/\sqrt{2}}) \quad (5.30)$$

We can now factor Equation (5.30) by combining like factors to obtain

$$E_{3,0}(x) = \tfrac{1}{4}(e^{x/\sqrt{2}}(e^{xi/\sqrt{2}} + e^{-xi/\sqrt{2}}) + e^{-x/\sqrt{2}}(e^{xi/\sqrt{2}} + e^{-xi/\sqrt{2}})) \quad (5.31)$$

$$E_{3,0}(x) = \frac{1}{4}((e^{x/\sqrt{2}} + e^{-x/\sqrt{2}})(e^{xi/\sqrt{2}} + e^{-xi/\sqrt{2}})) \quad (5.32)$$

$$E_{3,0}(x) = \cosh(\frac{x}{\sqrt{2}})\cos(\frac{x}{\sqrt{2}}) \quad (5.33)$$

Following the differentiation rule for $\psi_{mn}(x)$ given in Chapter 4, we obtain

$$E_{3,0}(x) = \cosh(\frac{x}{\sqrt{2}})\cos(\frac{x}{\sqrt{2}}) \quad (5.34)$$

$$E_{3,1}(x) = \frac{1}{\sqrt{2}}(\cosh(\frac{x}{\sqrt{2}})\sin(\frac{x}{\sqrt{2}}) + \sinh(\frac{x}{\sqrt{2}})\cos(\frac{x}{\sqrt{2}})) \quad (5.35)$$

$$E_{3,2}(x) = \sinh(\frac{x}{\sqrt{2}})\sin(\frac{x}{\sqrt{2}}) \quad (5.36)$$

$$E_{3,3}(x) = \frac{1}{\sqrt{2}}(\cosh(\frac{x}{\sqrt{2}})\sin(\frac{x}{\sqrt{2}}) - \sinh(\frac{x}{\sqrt{2}})\cos(\frac{x}{\sqrt{2}})) \quad (5.37)$$

(5.38)

As shown in Appendix C, for $m \geq 2$ the general rule is

$$E_{m,0}(x) = \frac{1}{\lceil 2^{m-3} \rceil} \sum_{p=0}^{\lceil 2^{m-3} \rceil - 1} \cosh(c_p x)\cos(s_p x) \quad (5.39)$$

Where we have used the short-hands

$$c_p \equiv \cos(\pi(2p+1)2^{1-m}) \tag{5.40}$$

$$s_p \equiv \sin(\pi(2p+1)2^{1-m}) \tag{5.41}$$

From $E_{m,0}(x)$, we can obtain $E_{mn}(x)$ for all n by repeated differentiation or integration. Alternatively, one can find $E_{mn}(x)$ from the following equation (see Appendix C)

$$\begin{aligned} E_{mn}(x) \quad = \quad & \frac{1}{\lceil 2^{m-1} \rceil} \sum_{p=0}^{\lceil 2^{m-1} \rceil - 1} i^{-n(2p+1)2^{m-2}} \cdot \\ & (\cosh(c_p x)\cos(s_p x) + \sinh(c_p x)\cos(s_p x) + \\ & i\cosh(c_p x)\sin(s_p x) + i\sinh(c_p x)\sin(s_p x)) \end{aligned} \tag{5.42}$$

For concreteness, the algebraic view of the Cairns functions for $m \leq 4$ are given in Appendix G.

Recall from Chapter 4 that $\psi_{mn}(x)$ is symmetric if n is even, and anti-symmetric if n is odd. The same rule must hold for $E_{mn}(x)$, since $E_{mn}(x) = \psi_{mn}(x)$. From the $E_{mn}(x)$ perspective, the symmetry rules hold because $\cosh(x)$ and $\cos(x)$ are symmetric functions, and $\sinh(x)$ and $\sin(x)$ are antisymmetric. The product of two symmetric functions or two anti-symmetric functions is symmetric, and the product of a symmetric with an anti-symmetric function is anti-symmetric. For even n, $E_{mn}(x)$ is a sum of products of the (symmetric) $\cosh(x)\cos(x)$ and $\sinh(x)\sin(x)$; for odd n, $E_{mn}(x)$ is a sum of products of the (anti-symmetric) $\cosh(x)\sin(x)$ and $\sinh(x)\cos(x)$.

Chapter 6

Projection Onto Cairns Space

The simplest of all infinite Taylor series is

$$e^x = \sum_{t=0}^{\infty} \frac{x^t}{t!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \tag{6.1}$$

By inspection of Equation (4.21), we see that every $\psi_{mn}(x)$ can be derived from the Taylor series for e^x by taking the terms of e^x and multiplying each of them by either -1, 0, or 1. More explicitly, we can build the below table, in which each entry gives the coefficient of a particular $\psi_{mn}(x)$ for a particular term in e^x.

	1	x	$\frac{x^2}{2!}$	$\frac{x^3}{3!}$	$\frac{x^4}{4!}$	$\frac{x^5}{5!}$	$\frac{x^6}{6!}$	$\frac{x^7}{7!}$	$\dots$
$\psi_{0,0}(x) = e^x$	1	1	1	1	1	1	1	1	$\dots$
$\psi_{1,0}(x) = e^{-x}$	1	-1	1	-1	1	-1	1	-1	$\dots$
$\psi_{2,0}(x) = \cos(x)$	1	0	-1	0	1	0	-1	0	$\dots$
$\psi_{2,1}(x) = \sin(x)$	0	1	0	-1	0	1	0	-1	$\dots$
$\psi_{3,0}(x)$	1	0	0	0	-1	0	0	0	$\dots$
$\psi_{3,1}(x)$	0	1	0	0	0	-1	0	0	$\dots$
$\psi_{3,2}(x)$	0	0	1	0	0	0	-1	0	$\dots$
$\psi_{3,3}(x)$	0	0	0	1	0	0	0	-1	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

On examining this table, one is struck by a very regular pattern. If we interpret the entries in each row as components of a vector in Euclidean space, it is fairly evident that the vectors are orthogonal.

To formalize this idea, let us define the vectors corresponding to the above table as follows. We take a finite approximation to Cairns space, where we consider only the first M levels and the first $L \equiv 2^M$ terms from functions in each of these levels. We recover the full Cairns space in the limit as $M \to \infty$.

$$\vec{v}_{m,n,L}(t) = \begin{cases} (-1)^{t \div 2^{m-1}} & \text{if } n = t \mod 2^{m-1} \text{ and } 0 \le t < L; \\ 0 & \text{Otherwise.} \end{cases} \tag{6.2}$$

for

$$\begin{aligned} M &\geq 0 && (6.3)\\ L &= 2^M && (6.4)\\ m &= 0, 1, \ldots, M && (6.5)\\ n &= 0, 1, \ldots, \lceil 2^{m-1} \rceil - 1 && (6.6) \end{aligned}$$

As is proved in Appendix D, this set of vectors is orthogonal: *i.e.*,

$$\vec{v0}_{m_0,n_0,L} \circ \vec{v1}_{m_1,n_1,L} = 0 \qquad \text{Unless } m_0 = m_1 \text{ and } n_0 = n_1 \tag{6.7}$$

Notice that L is equal to the total number of Cairns functions through level M. More explicitly,

M	L	# of Functions	Functions
0	1	1	$\psi_{0,0}(x)$
1	2	2	$\psi_{0,0}(x), \psi_{1,0}(x)$
2	4	4	$\psi_{0,0}(x), \psi_{1,0}(x), \psi_{2,0}(x), \psi_{2,1}(x)$
3	8	8	$\psi_{0,0}(x), \psi_{1,0}(x), \psi_{2,0}(x), \psi_{2,1}(x), \psi_{3,0}(x), \psi_{3,1}(x), \psi_{3,2}(x), \psi_{3,3}(x)$
...	...	...	...

This is a quite useful feature. It tells us that the first L vectors as defined above form an orthogonal set covering all Taylor series with degree less than L.

Consequently, any Taylor polynomial can be reversibly projected onto (a finite approximation of) Cairns space simply by computing a series of L dot products. And since the vectors to be projected onto contain only $-1, 0$ and 1 entries, computing the dot products requires only addition and subtraction of the terms in the Taylor series, followed by a divide to scale for the length of the vector projected onto.

We can normalize the vectors defined above into an orthonormal basis set by dividing each by its length. Since the fraction of terms that are non-zero at level m is $\frac{1}{\lceil 2^{m-1} \rceil}$, the number of non-zero terms in $\vec{v}_{m,n,L}$ is $\frac{L}{\lceil 2^{m-1} \rceil}$. Since the non-zero terms are all either 1 or -1, the vector length is

$$||\vec{v}_{m,n,L}|| = \sqrt{L / \lceil 2^{m-1} \rceil} \tag{6.8}$$

and of course the normalized basis vectors are

$$\hat{v}_{m,n,L} = \frac{\vec{v}_{m,n,L}}{||\vec{v}_{m,n,L}||} \tag{6.9}$$

In this chapter, we have seen that any function defined by a Taylor series can be trivially and reversibly mapped onto the Cairns functions by orthogonal projection. Once this mapping is complete, the function can be viewed in terms of its hyperbolic and circular components, simply by switching to the algebraic ($E_{m,n}(x)$) interpretation.

Doing so provides a decomposition similar to Fourier analysis; but while Fourier analysis represents only the periodic properties of the source function, Cairns space provides a balanced treatment of both exponential and periodic characteristics.

It may also be of interest that after projecting onto Cairns space one can measure a curvature associated with a function, in the sense outlined in Section 8.2, by calculating the relative weight of the projection on higher m-levels.

Chapter 7

The Broom Theorems

While the $\psi_{m,n}(x)$ functions define linearly-independent vectors, in the sense described in Chapter 6, there are methods to move between them using non-linear transforms. I call the methods for doing so the "broom theorems", since they allow information to be "swept" to different parts of Cairns space. To sweep up or down through the levels of Cairns space requires imaginary factors. It is also possible to "sweep sideways" at a particular level, through use of a displacement.

Roughly speaking,

- To sweep from a lower level m_u to a higher level m_v, we pick a set of $\psi_{m_v,n}(x)$ that cover the same terms as a given $\psi_{m_u,n}(x)$, use an imaginary factor to "turn off" the sign alternation at level m_v, then pick the appropriate sign for each function at level m_v to build up the $\psi_{m_u,n}(x)$.
- To sweep from a higher level m_v to a lower level m_u, we pick the function at level m_u that covers the same terms. It will also cover other terms. We duplicate the function at level m_u and use imaginary factors to cancel out the terms we do not need.
- To move sideways at a given m-level, we expand a function in its Taylor series, then group it in terms of the other functions at the same level in such a way that the original function is spread over all functions at that level.

The formalizations are provided below. See Appendix E for proofs.

7.1 Sweeping Up

To move from a lower to a higher m-level, we have

$$
\begin{aligned}
\psi_{m_u,n}(x) &= \sum_{p=0}^{c-1} i^{(-p\lceil 2^{m_u-1}\rceil+n)/\lceil 2^{m_v-2}\rceil} \cdot \\
&\quad (-1)^{p\lceil 2^{m_u-1}\rceil} \cdot \\
&\quad \psi_{(m_v,t\lceil 2^{m_u-1}\rceil+n)}(xi^{(2^{2-m_v})}) \\
&\quad \text{Where } 0 \le m_u < m_v \text{ and } c \equiv 2^{m_v-1}/\lceil 2^{m_u-1}\rceil
\end{aligned} \tag{7.1}
$$

Here are examples, which can be checked by expanding their Taylor series.

- *Map* $\psi_{0,0}(x) = e^x$ *to level* $m = 1$. $m_u = 0$, $m_v = 1$, $n = 0$, $c = 1$. We have $\psi_{0,0}(x) = \psi_{1,0}(-x)$, which is equivalent to the statement that $e^x = e^{-(-x)}$.
- *Map* $\psi_{0,0}(x) = e^x$ *to level* $m = 2$. $m_u = 0$, $m_v = 2$, $n = 0$, $c = 2$. We have $\psi_{0,0}(x) = \psi_{2,0}(ix)+i^{-1}\psi_{2,1}(ix)$, which is equivalent to the statement $e^x = \cos(ix)+i^{-1}\sin(ix)$. This is in a sense the inverse of Euler's formula; it is also equivalent to the hyperbolic identity $e^x = \cosh(x) + \sinh(x)$.
- *Map* $\psi_{1,0}(x) = e^{-x}$ *to level* $m = 2$. $m_u = 1$, $m_v = 2$, $n = 0$, $c = 2$. We have $\psi_{1,0}(x) = \psi_{2,0}(ix)-i^{-1}\psi_{2,1}(ix)$, which is equivalent to the statement $e^{-x} = \cos(ix) + -i^{-1}\sin(ix)$.
- *Map* $\psi_{2,0}(x) = \cos x$ *to level* $m = 3$. $m_u = 2$, $m_v = 3$, $n = 0$, $c = 2$. We have $\psi_{2,1}(x) = i^{-1/2}\psi_{3,1}(i^{1/2}x) - i^{-3/2}\psi_{3,3}(i^{1/2}x)$.
- *Map* $\psi_{2,1}(x) = \sin x$ *to level* $m = 4$. $m_u = 2$, $m_v = 4$, $n = 1$, $c = 4$. We have

$$\psi_{2,1}(x) = i^{-1/4}\psi_{4,1}(i^{1/4}x)-i^{-3/4}\psi_{4,3}(i^{1/4}x)+i^{-5/4}\psi_{4,5}(i^{1/4}x)-i^{-7/4}\psi_{4,7}(i^{1/4}x)$$

.

7.2 Sweeping Down

$$
\begin{aligned}
\psi_{m_v,n}(x) &= \left(\frac{1}{c}\right)\sum_{p=0}^{c-1} i^{-n(2p+1)(2^{2-m_v})}\psi_{m_u,n}(xi^{(2p+1)(2^{2-m_v})}) \\
&\quad \text{Where } 0 \le m_u < m_v \text{ and } c \equiv 2^{m_v-1}/\lceil 2^{m_u-1}\rceil
\end{aligned} \tag{7.2}
$$

The $i^{(2^{2-m_v})}$ factor in the $\psi_{m_u,n}$ argument is designed to provide a sign alternation every 2^{m_v-1} terms, which corresponds to the sign alternation pattern for level m_v. The $(2p+1)$ factor is designed to cancel out the unwanted terms at level m_u. The coefficient of $i^{-n(2p+1)\left(2^{2-m_v}\right)}$ performs a shift to handle $n > 0$ correctly.

Notice that in the special case $m_u = 0$, Equation (7.2) reduces to Equation (5.10). From this viewpoint, Equation (5.10) is simply the special case where one sweeps down to level $m = 0$. The proof of Equation (7.2) is therefore similar to the proof of Equation (5.10), differing in the number of stages required.

Here are some examples.

- *Map* $\psi_{1,0}(x) = e^{-x}$ *to* $\psi_{0,0}(x) = e^{x}$. $m_v = 1$, $m_u = 0$, $n = 0$, $c = 1$. We have $\psi_{1,0}(x) = \psi_{0,0}(-x)$, which is equivalent to the statement that $e^{-x} = e^{(-x)}$.

- *Map* $\psi_{2,0}(x) = \cos(x)$ *to* $\psi_{0,0}(x)$. $m_v = 2$, $m_u = 0$, $n = 0$, $c = 2$. We have $\psi_{2,0}(x) = \frac{1}{2}\left(\psi_{0,0}(ix) + \psi_{0,0}(i^3x)\right)$, which is equivalent to the statement that $\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$, familiar from trigonometry.

- *Map* $\psi_{2,0}(x) = \cos(x)$ *to* $\psi_{1,0}(x)$. $m_v = 2$, $m_u = 1$, $n = 0$, $c = 2$. We have $\psi_{2,0}(x) = \frac{1}{2}\left(\psi_{1,0}(ix) + \psi_{1,0}(i^3x)\right)$, which is equivalent to the statement that $\cos(x) = \frac{1}{2}\left(e^{-(ix)} + e^{-(-ix)}\right)$, equivalent to the above.

- *Map* $\psi_{3,0}(x)$ *to* $\psi_{2,0}(x)$. $m_v = 3$, $m_u = 2$, $n = 0$, $c = 2$. We have $\psi_{3,0}(x) = \frac{1}{2}\left(\psi_{2,0}(i^{1/2}x) + \psi_{2,0}(i^{3/2}x)\right)$, which can be checked by expanding the corresponding Taylor series.

- *Map* $\psi_{3,0}(x)$ *to* $\psi_{0,0}(x)$. $m_v = 3$, $m_u = 0$, $n = 0$, $c = 4$. We have $\psi_{3,0}(x) = \frac{1}{4}\left(\psi_{0,0}(i^{1/2}x) + \psi_{0,0}(i^{3/2}x) + \psi_{0,0}(i^{5/2}x) + +\psi_{0,0}(i^{7/2}x)\right)$, which again can be checked by expanding the Taylor series.

- *Map* $\psi_{3,1}(x)$ *to* $\psi_{2,1}(x)$. $m_v = 3$, $m_u = 2$, $n = 1$, $c = 2$. We have $\psi_{3,1}(x) = \frac{1}{2}\left(i^{-1/2}\psi_{2,1}(i^{1/2}x) + i^{-3/2}\psi_{2,1}(i^{3/2}x)\right)$, again checkable from the Taylor series.

7.3 Sweeping Sideways

We can spread any function at level m over all of the functions at level m by introducing a displacement. The general rule is as follows (see Appendix E for a proof)

$$\psi_{m,n}(x) = \sum_{p=0}^{\lceil 2^{m-1}\rceil - 1} \text{whole}(n-p)\psi_{m,p}(x-a)\psi_{m,(n-p) mod \lceil 2^{m-1}\rceil - 1}(a) \quad (7.3)$$

Where we define whole$(n-p) = 1$ for $(n-p) \geq 0$ and -1 otherwise.[1]

[1] whole is slightly different from the standard sgn function in that whole returns 1 if $(n-p) = 0$, whereas sgn returns 0.

For instance,

- *For $m = 0$, $n = 0$*, we have $\psi_{0,0}(x) = \psi_{0,0}(x-a)\psi_{0,0}(a)$, which in more familiar notation reads $e^x = e^{x-a}e^a$.
- *For $m = 1$, $n = 0$*, we have $\psi_{1,0}(x) = \psi_{1,0}(x-a)\psi_{1,0}(a)$, which is equivalent to $e^{-x} = e^{-(x-a)}e^{-a}$.
- *For $m = 2$, $n = 0$* we have $\psi_{2,0}(x) = \psi_{2,0}(x-a)\psi_{2,0}(a) - \psi_{2,1}(x-a)\psi_{2,1}(a)$, or equivalently $\cos(x) = \cos(x-a)\cos(a) - \sin(x-a)\sin(a)$. With the change of variables $u = x - a$, $v = a$, this is equivalent to the familiar double-angle formula $\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$.
- *For $m = 2$, $n = 1$* we have $\psi_{2,1}(x) = \psi_{2,0}(x-a)\psi_{2,1}(a) + \psi_{2,1}(x-a)\psi_{2,0}(a)$, or equivalently $\sin(x) = \cos(x-a)\sin(a) + \sin(x-a)\cos(a)$, which again is a familiar double-angle formula.
- *For $m = 3$, $n = 0$* we have

$$\psi_{3,0}(x) = \psi_{3,0}(x-a)\psi_{3,0}(a) - \psi_{3,1}(x-a)\psi_{3,3}(a) - \psi_{3,2}(x-a)\psi_{3,2}(a) - \psi_{3,3}(x-a)\psi_{3,1}(a)$$

Chapter 8

Sundry Findings

This chapter collects findings that are either not sufficiently important, or not sufficiently developed, to merit their own chapters.

8.1 Pythagorean Identities

Consider the well-known identity $\cos^2(x) + \sin^2(x) = 1$. This can be derived from Euler's formula by comparing

$$e^{ix}e^{-ix} = e^0 = 1 \tag{8.1}$$

With

$$\begin{aligned} e^{ix}e^{-ix} &= (\cos(x) + i\sin(x))(\cos(-x) + i\sin(-x)) && (8.2)\\ &= (\cos(x) + i\sin(x))(\cos(x) - i\sin(x)) && (8.3)\\ &= \cos(x)\cos(x) + \sin(x)\sin(x) && (8.4)\\ &= \cos^2(x) + \sin^2(x) && (8.5) \end{aligned}$$

Since the left sides of Equations (8.1) and (8.5) are equal, the right sides must also be equal: hence, $\cos^2(x) + \sin^2(x) = 1$.

Precisely the same procedure can be completed for $m = 3$, although the algebra is slightly more complicated. One finds

$$e^{xi^{1/2}}e^{-xi^{1/2}} = e^0 = 1 \tag{8.6}$$

And

$$e^{xi^{1/2}}e^{-xi^{1/2}} = \Big(\psi_{3,0}(x) + i^{\frac{1}{2}}\psi_{3,1}(x) + i\psi_{3,2}(x) + i^{\frac{3}{2}}\psi_{3,3}(x)\Big) \cdot \Big(\psi_{3,0}(x) - i^{\frac{1}{2}}\psi_{3,1}(x) + i\psi_{3,2}(x) - i^{\frac{3}{2}}\psi_{3,3}(x)\Big) \tag{8.7}$$

After multiplying through and canceling terms, we have

$$1 = (\psi_{3,0}^2(x) - \psi_{3,2}^2(x) + 2\psi_{3,1}(x)\psi_{3,3}(x)) + \tag{8.8}$$
$$i(\psi_{3,3}^2(x) - \psi_{3,1}^2(x) + 2\psi_{3,0}(x)\psi_{3,2}(x)) \tag{8.9}$$

Since the left-side of the above has no imaginary part, we have the two identities

$$1 = \psi_{3,0}^2(x) - \psi_{3,2}^2(x) + 2\psi_{3,1}(x)\psi_{3,3}(x) \tag{8.10}$$
$$0 = \psi_{3,3}^2(x) - \psi_{3,1}^2(x) + 2\psi_{3,0}(x)\psi_{3,2}(x) \tag{8.11}$$

The differential relationships between the $\psi_{3,n}(x)$ allow us the option to write these identities equivalently as

$$1 = \psi_{3,0}(x)\psi\prime_{3,1}(x) - \psi_{3,1}(x)\psi\prime_{3,0}(x) + \psi_{3,3}(x)\psi\prime_{3,2}(x) - \psi_{3,2}(x)\psi\prime_{3,3}(x) \tag{8.12}$$

$$0 = \psi_{3,0}(x)\psi\prime_{3,3}(x) - \psi_{3,3}(x)\psi\prime_{3,0}(x) + \psi_{3,2}(x)\psi\prime_{3,1}(x) - \psi_{3,1}(x)\psi\prime_{3,2}(x) \tag{8.13}$$

Which can be described in determinant form, if desired.

$$1 = \begin{vmatrix} \psi_{3,0}(x) & \psi_{3,1}(x) \\ \psi\prime_{3,0}(x) & \psi\prime_{3,1}(x) \end{vmatrix} - \begin{vmatrix} \psi_{3,2}(x) & \psi_{3,3}(x) \\ \psi\prime_{3,2}(x) & \psi\prime_{3,3}(x) \end{vmatrix}$$

and

$$0 = \begin{vmatrix} \psi_{3,0}(x) & \psi_{3,3}(x) \\ \psi\prime_{3,0}(x) & \psi\prime_{3,3}(x) \end{vmatrix} - \begin{vmatrix} \psi_{3,1}(x) & \psi_{3,2}(x) \\ \psi\prime_{3,1}(x) & \psi\prime_{3,2}(x) \end{vmatrix}$$

Yet another way of representing these identities involves the natural logarithm.

$$\begin{aligned}
1 &= -(\psi_{3,1}(x)\psi\prime_{3,0}(x) - \psi_{3,0}(x)\psi\prime_{3,1}(x)) + \\
&\quad (\psi_{3,3}(x)\psi\prime_{3,2}(x) - \psi_{3,2}(x)\psi\prime_{3,3}(x)) && (8.14)\\
&= -\frac{d}{dx}\left(\frac{\psi_{3,0}(x)}{\psi_{3,1}(x)}\right)\psi^2_{3,1}(x) + \frac{d}{dx}\left(\frac{\psi_{3,2}(x)}{\psi_{3,3}(x)}\right)\psi^2_{3,3}(x) && (8.15)\\
&= -\frac{d^2}{dx^2}\left(\ln(\psi_{3,1}(x))\right)\psi^2_{3,1}(x) + \frac{d^2}{dx^2}\left(\ln(\psi_{3,3}(x))\right)\psi^2_{3,3}(x) && (8.16)
\end{aligned}$$

And similarly

$$0 = -\frac{d^2}{dx^2}\left(\ln(\psi_{3,0}(x))\right)\psi^2_{3,0}(x) + \frac{d^2}{dx^2}\left(\ln(\psi_{3,2}(x))\right)\psi^2_{3,2}(x)$$

These relationships exist in simpler form at $m = 2$ with $\psi_{2,0}(x) = \cos(x)$ and $\psi_{2,1}(x) = \sin(x)$. For instance,

$$\begin{aligned}
1 &= \cos^2(x) + \sin^2(x) && (8.17)\\
&= \psi^2_{2,0}(x) + \psi^2_{2,1}(x) && (8.18)\\
&= \psi_{2,0}(x)\psi\prime_{2,1}(x) + \psi_{2,1}(x)(-\psi\prime_{2,0}(x)) && (8.19)\\
&= \psi_{2,0}(x)\psi\prime_{2,1}(x) - \psi_{2,1}(x)\psi\prime_{2,0}(x) && (8.20)
\end{aligned}$$

The same procedure can be pursued for higher m-levels.

8.2 Curvature

Equation (3.8) can be thought of as defining a triangle with angle $x\sin(\pi 2^{1-m})$ and magnitude $e^{x\cos(\pi 2^{1-m})}$.

Let us define

$$\begin{aligned}
\theta_m &\equiv x\sin(\pi 2^{1-m}) && (8.21)\\
L_m &\equiv e^{x\cos(\pi 2^{1-m})} && (8.22)
\end{aligned}$$

For fixed x, as m increases θ_m decreases and L_m increases. The triangle becomes longer and narrower.

We can also form a triangle by adding across m-levels the real and imaginary parts of $e^{xi^{\left(2^{2-m}\right)}}$. If we do so, for positive x the larger m-levels will dominate the summation as x increases, since they grow more rapidly with x. This implies that as x increases, the composite triangle will become more narrow; or, put differently, the angle between the sides of the triangle depends on the length of the sides. This is a characteristic of non-Euclidean geometry.

For negative x, as the magnitude of x increases the relative influence of the higher m-levels will decline, because they decrease more rapidly with negative x.

If we view from the perspective of (say) level $m = 5$, the composite triangle will appear to narrow with increasing positive x, and widen with decreasing negative x. The former corresponds to elliptical geometry, the latter to hyperbolic geometry.

Note that $m = 2$ provides a perfectly Euclidean geometry.

8.3 The Inner Product on Level m

It is possible to compute the inner product at level m with the following equation (see Appendix F for a derivation).

$$\begin{aligned} & E_{m,n}(x) \circ E_{m,n}(x+\alpha) \qquad (8.23) \\ & \quad \equiv \sum_{n=0}^{\lceil 2^{m-1} \rceil -1} E_{m,n}(x) E_{m,n}(x+\alpha) \\ & \quad = \frac{1}{\lceil 2^{m-1} \rceil} \sum_{n=0}^{\lceil 2^{m-1} \rceil -1} e^{2x \cos(\pi(2p+1)2^{1-m})} e^{\alpha i^{-(2p+1)2^{2-m}}} \end{aligned}$$

Here are some examples.

- For $m = 0$, Equation (8.24) reduces to

$$\begin{aligned} E_{0,0}(x)E_{0,0}(x+\alpha) &= e^x e^{x+\alpha} && (8.24) \\ &= e^{2x\cos(2\pi)} e^{\alpha i^{-4}} && (8.25) \\ &= e^{2x} e^{\alpha} && (8.26) \\ &= e^{2x+\alpha} && (8.27) \end{aligned}$$

which is simply the familiar rule for adding exponents.

- For $m = 1$, Equation (8.24) reduces to

$$
\begin{aligned}
E_{1,0}(x)E_{1,0}(x+\alpha) &= e^{-x}e^{-x-\alpha} && (8.28)\\
&= e^{2x\cos(\pi)}e^{\alpha i^{-2}} && (8.29)\\
&= e^{-2x}e^{-\alpha} && (8.30)\\
&= e^{-2x-\alpha} && (8.31)
\end{aligned}
$$

  which is the rule for adding negative exponents.

- For $m = 2$, Equation (8.24) reduces to

$$
\begin{aligned}
&E_{2,0}(x)E_{2,0}(x+\alpha) + E_{2,1}(x)E_{2,1}(x+\alpha) && (8.32)\\
&\quad = \cos(x)\cos(x+\alpha) + \sin x \sin(x+\alpha) && (8.33)\\
&\quad = \frac{1}{2}\left(e^{2x\cos(\pi/2)}e^{\alpha i^{-1}} + e^{2x\cos(3\pi/2)}e^{\alpha i^{-3}}\right) && (8.34)\\
&\quad = \frac{1}{2}\left(e^{0}e^{\alpha(-i)} + e^{0}e^{\alpha i}\right) && (8.35)\\
&\quad = \cos(\alpha) && (8.36)
\end{aligned}
$$

  This is also familiar, as the dot product of two vectors in a two-dimensional Euclidean plane.

For $m \geq 3$ the inner product becomes more complex, but no less interesting.[1]

Notice that only for $m = 2$, corresponding to Euler's formula, is the inner product independent of x. This is to be expected:[2] only for $m = 2$ does the suitcase term $e^{xi^{\left(2^{2-m}\right)}}$ correspond to a perfect circle, with no growth term.

8.4 Differentiation, Rotation and Displacement

Differentiation (or integration), imaginary rotation and displacement are closely related in Cairns space. We can write

$$
\begin{aligned}
\frac{d}{dx}e^{xi^{\left(2^{2-m}\right)}} &= i^{\left(2^{2-m}\right)}e^{xi^{\left(2^{2-m}\right)}} && (8.37)\\
&= e^{i\pi 2^{1-m}}e^{xi^{\left(2^{2-m}\right)}} && (8.38)\\
&= e^{i^{\left(2^{2-m}\right)}\left(x+i\pi 2^{1-m}i^{-\left(2^{2-m}\right)}\right)} && (8.39)
\end{aligned}
$$

[1] Although notably more difficult to typeset.
[2] In retrospect, I admit.

For conciseness, define the displacement

$$\Delta_m \equiv i\pi 2^{1-m} i^{-(2^{2-m})} \tag{8.40}$$

Then

$$\frac{d}{dx} e^{xi^{\left(2^{2-m}\right)}} = i^{(2^{2-m})} e^{xi^{\left(2^{2-m}\right)}} = e^{(x+\Delta_m)i^{\left(2^{2-m}\right)}} \tag{8.41}$$

So we can think of differentiation as either an imaginary multiplication (which implies a rotation in the complex plane) or as a shift in the value of x by Δ_m.[3] Δ_m is a characteristic value of a given m-level. We find

- *For* $m = 0$, $\Delta_0 = 2\pi i$;
- *For* $m = 1$, $\Delta_1 = -\pi i$;
- *For* $m = 2$, $\Delta_2 = \pi/2$;
- *For* $m = 3$, $\Delta_3 = \pi i^{1/2}/4$;
- *For* $m = 4$, $\Delta_4 = \pi i^{3/4}/8$

For instance, for $m = 1$ we have

$$\frac{d}{dx} e^{-x} = e^{-(x+\Delta_m)} = e^{-x+\pi i} = e^{-x} e^{\pi i} = e^{-x}(-1) = -e^{-x}$$

Since

$$e^{xi^{\left(2^{2-m}\right)}} = \sum_{n=0}^{\lceil 2^{m-1}\rceil -1} i^{n2^{2-m}} E_{mn}(x) \tag{8.42}$$

We can equate the two views of differentiation to get

$$\sum_{n=0}^{\lceil 2^{m-1}\rceil -1} i^{n2^{2-m}} E_{mn}(x+\Delta_m) = \sum_{n=0}^{\lceil 2^{m-1}\rceil -1} i^{(n+1)2^{2-m}} E_{mn}(x) \tag{8.43}$$

[3]To integrate instead of differentiate, simply reverse the exponent sign of $i^{(2^{2-m})}$ to $i^{-(2^{2-m})}$, and reverse the sign of Δ_m.

Note, however, that the above is a statement about the summation as a whole, not about particular terms in the summation. In particular, for $m \geq 3$ it is *not* true that[4]

$$E_{m,n<\lceil 2^{m-1}\rceil -1}(x) \stackrel{?}{=} E_{m,n+1}(x+\Delta_m) \quad \textit{False!} \tag{8.44}$$

However, Equation (8.44) *is* true for $m = 2$, where it reduces to the familiar rule that

$$\cos(x) = \sin(x + \pi/2)$$

8.5 The Roots of n$^{\text{th}}$-Degree Polynomials

It is a singular fact that the equation

$$a\cos(x) - b\sin(x) = 0 \tag{8.45}$$

can be solved simply by rotation.[5]

To see this, note that if a and b are not both zero, we can write the above as

$$L\left(\frac{a}{L}\cos(x) - \frac{b}{L}\sin(x)\right) = 0 \tag{8.46}$$

for $L \equiv \sqrt{a^2 + b^2}$.

$\frac{a}{L}$ and $\frac{b}{L}$ are now in the form of the cos and sin of some angle α, defined by

$$\cos(\alpha) = \frac{a}{L} \Rightarrow \alpha = \arccos(\frac{a}{L}) \tag{8.47}$$

Or equivalently

$$\sin(\alpha) = \frac{b}{L} \Rightarrow \alpha = \arcsin(\frac{b}{L}) \tag{8.48}$$

α is the angle formed by a right triangle with sides a and b and hypotenuse L.

This gives us

$$L\left(\cos(\alpha)\cos(x) - \sin(\alpha)\sin(x)\right) = 0 \tag{8.49}$$

[4] see Section 7.3 for the general relationship between functions at a given level.

[5] Taking the second term to be negative is of course arbitrary, but useful below.

Or

$$L\cos(\alpha + x) = 0 \tag{8.50}$$

This is a remarkable relationship. It says that any sum of a cos and sin function with the same period is equal to a single shifted, scaled cos function.

And while it is not clear for what values of x

$$a\cos(x) - b\sin(x) = 0 \tag{8.51}$$

is true,

$$L\cos(\alpha + x) = 0 \tag{8.52}$$

obviously holds when $x = \pm\pi/2 - \alpha$. The roots are found simply by rotating by $\pm\pi/2$ from $-\alpha$.

Does a generalization of the above hold for Cairns space generally? If so, one could imagine solving for the roots of polynomials by projecting onto Cairns space and performing rotations.

Here is an intriguing special case. From Equation (3.8) we know that the real and imaginary parts of $e^{xi^{\left(2^{2-m}\right)}}$ are respectively

$$R_m(x) = e^{x\cos(\pi 2^{1-m})}\cos(x\sin(\pi 2^{1-m})) \tag{8.53}$$
$$I_m(x) = e^{x\cos(\pi 2^{1-m})}\sin(x\sin(\pi 2^{1-m})) \tag{8.54}$$

and therefore we have the ratio

$$\frac{I_m(x)}{R_m(x)} = \tan(x\sin(\pi 2^{1-m})) \tag{8.55}$$

From Equation (5.12) we know that $E_{m,n}(x)$ has associated with it a coefficient of $i^{n\left(2^{2-m}\right)}$. We may split this into real and imaginary components as follows.

$$i^{n\left(2^{2-m}\right)} = e^{i\pi n\left(2^{1-m}\right)} = \cos(\pi n 2^{1-m}) + i\sin(\pi n 2^{1-m}) \tag{8.56}$$

and therefore, from Equation (5.12), we have

$$R_m(x) = \sum_{n=0}^{\lceil 2^{m-1}\rceil -1} \cos(\pi n 2^{1-m}) E_{mn}(x) \tag{8.57}$$

$$I_m(x) = \sum_{n=0}^{\lceil 2^{m-1}\rceil -1} \sin(\pi n 2^{1-m}) E_{mn}(x) \tag{8.58}$$

In a sense, $\cos(\pi n 2^{1-m})$ and $\sin(\pi n 2^{1-m})$ are the "natural" coefficients of $E_{m,n}(x)$ (or of $\psi_{m,n}(x)$). For any polynomial consisting of terms at level m with coefficients that are multiples of the natural coefficients, finding roots of the polynomial is trivial. More explicitly, let

$$\sum_{n=0}^{\lceil 2^{m-1}\rceil -1} a_n E_{mn}(x) = 0 \tag{8.59}$$

If there are k_R and k_I such that

$$a_n = k_R \cos(\pi n 2^{1-m}) + k_I \sin(\pi n 2^{1-m}) \tag{8.60}$$

then a solution is[6]

$$x = \arctan(\frac{-k_R}{k_I}) / \sin(\pi 2^{1-m}) \tag{8.61}$$

One can imagine using the broom theorems, or other means, to collect the projection of a polynomial onto a single level, then distribute it according to the natural coefficients. However, If there is a general solution to the problem of finding polynomial roots in Cairns space I have not found it as yet, and not for lack of trying. I will not describe here the techniques that do *not* work, for the enumeration is dismal; and the hand of time is fleeting, and my cigar is burning out.

[6] Incidentally, there is a "hyperbolic dual" in which one uses Equation (3.12) to break the suitcase into hyperbolic, rather than circular, components.

Chapter 9

Conclusion

But we are not conceived as angels are:
A thousand mindless cares our minds must mar,
Whose troubled tides, through time, turn tyrannous tsar.

In this monograph, the nine characters of the suitcase, $e^{xi^{\left(2^{2-m}\right)}}$, have been unpacked to some sixty pages; and it is clear that we have seen neither the sides nor the depths of the matter.

There is an interesting debate as to whether mathematics is invented or discovered. On my own experience, I must come down solidly on the side of discovered. I could not have invented the magic suitcase, simply because it is much smarter than I am.

I found what I would not have imagined, nor thought possible if imagined, nor known how to achieve if thought possible. I stumbled across things, very much like a child at the beach, or (in my better moments) like an archaeologist carefully excavating a buried palace. I learned mathematics by studying the suitcase; I taught it nothing.

I have a strong sense of mathematics as existing in a "place". A place very different from what we are used to, to be sure, but a place nonetheless. If one asks if it is a place as real as one on Earth, I would say "more real". It existed before our planet was formed, it will exist after the Earth is gone; it is independent of our universe itself. And it is independent of ourselves as well. No one can carve their initials in the sides of mathematics.

There is something glorious about learning mathematics, but a kind of sadness as well. I do not fully understand what the suitcase entails, and I very much suspect that I never will. I am conscious of dimly viewing a garden to which I can never find the key, and can never be admitted.

'Tis ever and of needs a sorrowful grace
To see a distant beauty, but in trace,
And know it is no work of mortal race.

Appendix A

Derivation of $\psi_{mn}(x)$

In Chapter 4, it was stated that by grouping terms with like coefficients of i one could write the suitcase as

$$e^{xi^{\left(2^{2-m}\right)}} = \sum_{n=0}^{\lceil 2^{m-1} \rceil - 1} i^{n2^{1-m}} \psi_{mn}(x) \tag{A.1}$$

where

$$\psi_{mn}(x) = \sum_{t=0}^{\infty} (-1)^{t\lceil 2^{1-m} \rceil} \frac{x^{t\lceil 2^{m-1} \rceil + n}}{(t\lceil 2^{m-1} \rceil + n)!} \tag{A.2}$$

This appendix proves this assertion, by a generalization of the procedure given in Chapter 4 for $m = 2$ and $m = 3$. We proceed by expanding $e^{xi^{\left(2^{2-m}\right)}}$ as a Taylor series and grouping terms.

$$e^{xi^{\left(2^{2-m}\right)}} \equiv \sum_{t=0}^{\infty} \frac{\left(xi^{\left(2^{2-m}\right)}\right)^t}{t!} \tag{A.3}$$

$$= \sum_{t=0}^{\infty} \frac{\left(x^t i^{\left(t2^{2-m}\right)}\right)}{t!} \tag{A.4}$$

Notice that

$$i^{\left(t2^{2-m}\right)} = -1$$

Occurs when

$$t2^{2-m} = 2 \Rightarrow t = 2 \cdot 2^{m-2} \quad (A.5)$$

$$= 2^{m-1} \quad (A.6)$$

After $t = 2^{m-1}$ terms the pattern of i exponents will repeat, with sign reversed. Let us call this the "step size".

Since we are interested in grouping terms with like factors of i, we need to collect terms that are separated by the step size. This gives us (roughly)

$$e^{xi^{\left(2^{2-m}\right)}} = \sum_{n=0}^{\lceil 2^{m-1}\rceil -1} i^{n\left(2^{2-m}\right)} \quad (A.7)$$

$$\cdot \sum_{t=n}^{\infty} (-1)^t \frac{x^{t2^{m-1}+n}}{(t2^{m-1}+n)!} \quad (A.8)$$

Essentially, we simply define $\psi_{mn}(x)$ by the second summation in the above equation. The one remaining issue is to handle the case $m \leq 0$ correctly, which corresponds to the function e^x.

Comparison with the Taylor series for e^x suggests that we change 2^{m-1} to $\lceil 2^{m-1} \rceil$. This prevents 2^{m-1} from becoming fractional for $m \leq 0$, and provides the correct behavior for e^x. (Of course, the ceiling functions have no effect for $m \geq 1$.)

The second issue for $m \leq 0$ is to get the sign right. For $m \leq 0$ the step size $t = 2^{m-1}$ is less than one; the meaning is that the sign never changes.[1] We can handle this special case adequately, if not entirely elegantly, by writing

$$\psi_{mn}(x) = \sum_{t=0}^{\infty} (-1)^{t\lceil 2^{1-m}\rceil} \frac{x^{t\lceil 2^{m-1}\rceil+n}}{(t\lceil 2^{m-1}\rceil + n)!} \quad (A.9)$$

For $m \geq 1$ the $\lceil 2^{1-m} \rceil$ factor is simply 1 and has no effect. For $m \leq 0$ this factor will be a power of two; since -1 raised to any power of two is 1, this has the effect of preventing a sign change between terms.[2]

[1] More precisely, it implies that the sign changes at least twice between adjacent terms, so that by the time one gets to the next term the sign is back where it was.

[2] In the software industry, this would be known as a "hack". I prefer the phrase "adequate, if not entirely elegant."

Appendix B

Proof of the Equivalence of $\psi_{mn}(x)$ and $E_{mn}(x)$

This appendix proves that as claimed by Equation (5.11), the Taylor series view of the components of $e^{xi^{\left(2^{2-m}\right)}}$, given by $\psi_{mn}(x)$, is equivalent to the algebraic view of the components given by $E_{mn}(x)$.

To review, these two views of the suitcase are defined by

$$\psi_{mn}(x) = \sum_{t=0}^{\infty} (-1)^{t\lceil 2^{1-m}\rceil} \frac{x^{t\lceil 2^{m-1}\rceil+n}}{(t\lceil 2^{m-1}\rceil+n)!} \tag{B.1}$$

$$E_{mn}(x) \equiv \frac{1}{\lceil 2^{m-1}\rceil} \sum_{p=0}^{\lceil 2^{m-1}\rceil-1} i^{-n(2p+1)2^{2-m}} e^{xi^{(2p+1)2^{2-m}}} \tag{B.2}$$

Note that it is sufficient to prove the case for $E_{m,0}(x) = \psi_{m,0}(x)$. Since the $\psi_{mn}(x)$ and $E_{mn}(x)$ can be defined by differentiation (or integration) of $\psi_{m,0}(x)$ and $E_{m,0}(x)$ respectively, the equivalence of $E_{m,0}(x)$ and $\psi_{m,0}(x)$ implies the equivalence of all other functions in their respective families.

The proof proceeds by expanding $E_{m,0}(x)$ as a sum of Taylor series, then recursively canceling terms until we are left with $\psi_{m,0}(x)$.

$$E_{m,0}(x) \equiv \frac{1}{\lceil 2^{m-1}\rceil} \sum_{p=0}^{\lceil 2^{m-1}\rceil-1} e^{xi^{(2p+1)2^{2-m}}} \tag{B.3}$$

$$= \frac{1}{\lceil 2^{m-1}\rceil} \sum_{p=0}^{\lceil 2^{m-1}\rceil-1} \sum_{t=0}^{\infty} \frac{\left(xi^{(2p+1)2^{2-m}}\right)^t}{t!} \tag{B.4}$$

where in the last step we expanded the exponential as a Taylor series. Next, split the series into 2^{m-1} overlapping summations, with index n. This gives us

$$E_{m,0}(x) = \frac{1}{\lceil 2^{m-1}\rceil} \sum_{n=0}^{\lceil 2^{m-1}\rceil -1} \sum_{p=0}^{\lceil 2^{m-1}\rceil -1} \sum_{t=0}^{\infty} \tag{B.5}$$

$$\frac{\left(xi^{(2p+1)2^{2-m}}\right)^{\lceil 2^{m-1}\rceil t+n}}{(\lceil 2^{m-1}\rceil t+n)!} \tag{B.6}$$

$$= \frac{1}{\lceil 2^{m-1}\rceil} \sum_{n=0}^{\lceil 2^{m-1}\rceil -1} \sum_{p=0}^{\lceil 2^{m-1}\rceil -1} \sum_{t=0}^{\infty} \tag{B.7}$$

$$\left[\left(i^{(2p+1)2^{2-m}}\right)^{\lceil 2^{m-1}\rceil t+n}\right] \frac{x^{\lceil 2^{m-1}\rceil t+n}}{(\lceil 2^{m-1}\rceil t+n)!} \tag{B.8}$$

For $n = 0$, the term in square brackets becomes

$$\sum_{p=0}^{\lceil 2^{m-1}\rceil -1} \left[i^{2t} i^{4pt}\right] = \sum_{p=0}^{\lceil 2^{m-1}\rceil -1} \left[(-1)^t (1)^{pt}\right] \tag{B.9}$$

$$= \lceil 2^{m-1}\rceil (-1)^t \tag{B.10}$$

Consequently, for $n = 0$ we have $\psi_{m,0}(x) = E_{m,0}(x)$. Therefore, to prove $\psi_{m,0}(x) = E_{m,0}(x)$ it is necessary and sufficient to prove that the terms in square brackets are always zero for $n \geq 1$.

The terms in square brackets can be written as

$$\sum_{p=0}^{\lceil 2^{m-1}\rceil -1} i^{2t} i^{4pt} i^{n2^{2-m}} i^{pn2^{3-m}} = \sum_{p=0}^{\lceil 2^{m-1}\rceil -1} (-1)^t i^{n(2p+1)2^{2-m}} \tag{B.11}$$

Next, split the above summation into two parts, essentially comparing the first and second quadrants of the complex plane to the third and fourth quadrants. We obtain

$$\sum_{p=0}^{\lceil 2^{m-2}\rceil -1} (-1)^t i^{n(2p+1)(2^{2-m})} + \sum_{p=\lceil 2^{m-2}\rceil}^{\lceil 2^{m-1}\rceil -1} (-1)^t i^{n(2p+1)(2^{2-m})} \tag{B.12}$$

Reparameterizing the second sum gives

$$\sum_{p=0}^{\lceil 2^{m-2}\rceil -1} (-1)^t i^{n(2p+1)(2^{2-m})} + \sum_{p=\lceil 2^{m-2}\rceil}^{\lceil 2^{m-1}\rceil -1} (-1)^t i^{n(2p+1)(2^{2-m})} \tag{B.13}$$

$$= \sum_{p=0}^{\lceil 2^{m-2}\rceil -1} (-1)^t i^{n(2p+1)(2^{2-m})} \tag{B.14}$$

$$+ \sum_{p=0}^{\lceil 2^{m-2}\rceil -1} (-1)^t i^{n(2[p+2^{m-2}]+1)(2^{2-m})} \tag{B.15}$$

$$= \sum_{p=0}^{\lceil 2^{m-2}\rceil -1} (-1)^t i^{n(2p+1)(2^{2-m})} \tag{B.16}$$

$$+ \sum_{p=0}^{\lceil 2^{m-2}\rceil -1} (-1)^t i^{n(2p+1)(2^{2-m})} i^{2n} \tag{B.17}$$

$$= \sum_{p=0}^{\lceil 2^{m-2}\rceil -1} (-1)^t i^{n(2p+1)(2^{2-m})} \tag{B.18}$$

$$+ \sum_{p=0}^{\lceil 2^{m-2}\rceil -1} (-1)^t i^{n(2p+1)(2^{2-m})} (-1)^n \tag{B.19}$$

The two summations differ only by a factor of $(-1)^n$: consequently, the sum will be zero for any odd n. For even n, the two sums are equal, so we can rewrite the summation as

$$\sum_{p=0}^{\lceil 2^{m-2}\rceil -1} 2(-1)^t i^{n(2p+1)} 2^{2-m} \tag{B.20}$$

We can again split the summation in two, to compare the first quadrant to the second quadrant. We find

$$\sum_{p=0}^{\lceil 2^{m-3}\rceil -1} (-1)^t i^{n(2p+1)(2^{2-m})} + \sum_{p=\lceil 2^{m-3}\rceil}^{\lceil 2^{m-2}\rceil -1} (-1)^t i^{n(2p+1)(2^{2-m})} \tag{B.21}$$

Reparameterizing the second summation gives

$$\sum_{p=0}^{\lceil 2^{m-3}\rceil -1} (-1)^t i^{n(2p+1)(2^{2-m})} + \sum_{p=0}^{\lceil 2^{m-3}\rceil -1} -1^t i^{n(2[p+2^{m-2}]+1)(2^{2-m})} \tag{B.22}$$

$$= \sum_{p=0}^{\lceil 2^{m-3}\rceil -1} (-1)^t i^{n(2p+1)(2^{2-m})} + \sum_{p=0}^{\lceil 2^{m-3}\rceil -1} (-1)^t i^{n(2p+1)(2^{2-m})} i^n \tag{B.23}$$

If n contains only a single factor of 2, then $i^n = -1$ and the two summations again cancel. Since n cannot be odd and cannot contain a single factor of n, it must contain a factor of 4, in which case the two summations are equal to each other.

We can repeat this process as many times as necessary to eliminate the possibility of any $n > 0$ producing a non-zero sum. In general, this will take $m-1$ steps, since there $\lceil 2^{m-1} \rceil$possible values of n and each step eliminates half of the remaining possibilities.

Appendix C

Factoring $E_{mn}(x)$

This appendix derives Equations (5.39) and (5.42), relating to the factors of $E_{m,n}(x)$.

C.1 Derivation of $E_{m,0}$ Factors

This section proves Equation (5.39):

$$E_{m,0}(x) = \frac{1}{\lceil 2^{m-3} \rceil} \sum_{p=0}^{\lceil 2^{m-3} \rceil - 1} \cosh(c_p x) \cos(s_p x) \tag{C.1}$$

Where we have used the short-hands introduced in Chapter 5:

$$c_p \equiv \cos(\pi(2p+1)2^{1-m}) \tag{C.2}$$
$$s_p \equiv \sin(\pi(2p+1)2^{1-m}) \tag{C.3}$$

Equation (C.1) can be derived by a generalization of the procedure given in Chapter 5 for $m = 3$.

By definition (see Equation (5.9)) we have

$$E_{m,0}(x) \equiv \frac{1}{\lceil 2^{m-1} \rceil} \sum_{p=0}^{\lceil 2^{m-1} \rceil - 1} e^{xi^{(2p+1)2^{2-m}}} \tag{C.4}$$

We can split this summation in two, giving

$$E_{m,0}(x) = \frac{1}{\lceil 2^{m-1} \rceil} \left(\sum_{p=0}^{\lceil 2^{m-2} \rceil - 1} e^{xi^{(2p+1)2^{2-m}}} + \sum_{p=\lceil 2^{m-2} \rceil}^{\lceil 2^{m-1} \rceil - 1} e^{xi^{(2p+1)2^{2-m}}} \right) \tag{C.5}$$

The general idea is to build $\cos(x)$ terms by combining terms between the two summations. We will then repeat the process to build the $\cosh(x)$ terms. To begin, we first reparameterize the second summation.

$$\sum_{p=\lceil 2^{m-2}\rceil}^{\lceil 2^{m-1}\rceil-1} e^{xi^{(2p+1)2^{2-m}}} = \sum_{p=0}^{\lceil 2^{m-2}\rceil-1} e^{xi^{(2(p+\lceil 2^{m-2}\rceil)+1)2^{2-m}}} \tag{C.6}$$

$$= \sum_{p=0}^{\lceil 2^{m-2}\rceil-1} e^{xi^{(2p+1)2^{2-m}}i^2} \tag{C.7}$$

Next, we reverse the direction of the second summation, by replacing p with $\lceil 2^{m-2}\rceil - 1 - p$. This is important because we will want to group the first terms of the first summation with the last terms of the second summation.

$$\sum_{p=0}^{\lceil 2^{m-2}\rceil-1} e^{xi^{(2p+1)2^{2-m}}i^2} = \sum_{p=0}^{\lceil 2^{m-2}\rceil-1} e^{xi^{(2(\lceil 2^{m-2}\rceil-1-p)+1)2^{2-m}}i^2} \tag{C.8}$$

Noting that $i^2 = i^{(2^{m-1}2^{2-m})}$, we can write

$$\sum_{p=0}^{\lceil 2^{m-2}\rceil -1} e^{xi^{(2(\lceil 2^{m-2}\rceil -1-p)+1)2^{2-m}} i^2} \tag{C.9}$$

$$= \sum_{p=0}^{\lceil 2^{m-2}\rceil -1} e^{xi^{(2^{m-1}+2(\lceil 2^{m-2}\rceil -1-p)+1)2^{2-m}}} \tag{C.10}$$

$$= \sum_{p=0}^{\lceil 2^{m-2}\rceil -1} e^{xi^{(2^{m-1}+(2^{m-1}-2-2p)+1)2^{2-m}}} \tag{C.11}$$

$$= \sum_{p=0}^{\lceil 2^{m-2}\rceil -1} e^{xi^{(2^{m}-1-2p)2^{2-m}}} \tag{C.12}$$

$$= \sum_{p=0}^{\lceil 2^{m-2}\rceil -1} e^{xi^4 i^{(-2p-1)2^{2-m}}} \tag{C.13}$$

$$= \sum_{p=0}^{\lceil 2^{m-2}\rceil -1} e^{xi^{-(2p+1)2^{2-m}}} \tag{C.14}$$

Next, plug back into Equation (C.5) to get

$$E_{m,0}(x) = \frac{1}{\lceil 2^{m-1}\rceil}\left(\sum_{p=0}^{\lceil 2^{m-2}\rceil -1} e^{xi^{(2p+1)2^{2-m}}} + \sum_{p=0}^{\lceil 2^{m-2}\rceil -1} e^{xi^{-(2p+1)2^{2-m}}}\right) \tag{C.15}$$

As in Chapter 3, we can use the equation

$$i = e^{i\pi/2}$$

to simplify the summation further. We have

$$
\begin{aligned}
&e^{xi^{(2p+1)2^{2-m}}} + e^{xi^{-(2p+1)2^{2-m}}} \\
&\quad = e^{xe^{(i\pi(2p+1)2^{2-m})/2}} + e^{xe^{-(i\pi(2p+1)2^{2-m})/2}} \\
&\quad = e^{x\left(\cos(\pi(2p+1)2^{1-m})+i\sin(\pi(2p+1)2^{1-m})\right)} + e^{x\left(\cos(-\pi(2p+1)2^{1-m})+i\sin(-\pi(2p+1)2^{1-m})\right)} \\
&\quad = e^{x\left(\cos(\pi(2p+1)2^{1-m})+i\sin(\pi(2p+1)2^{1-m})\right)} + e^{x\left(\cos(\pi(2p+1)2^{1-m})-i\sin(\pi(2p+1)2^{1-m})\right)} \\
&\quad = e^{x\cos(\pi(2p+1)2^{1-m})}\left(e^{ix\sin(\pi(2p+1)2^{1-m})} + e^{ix\sin(-\pi(2p+1)2^{1-m})}\right) \\
&\quad = e^{x\cos(\pi(2p+1)2^{1-m})}(\cos(x\sin(\pi(2p+1)2^{1-m})) + i\sin(x\sin(\pi(2p+1)2^{1-m})) \\
&\quad + \cos(x\sin(-\pi(2p+1)2^{1-m})) + i\sin(x\sin(-\pi(2p+1)2^{1-m}))) \\
&\quad = e^{x\cos(\pi(2p+1)2^{1-m})}(\cos(x\sin(\pi(2p+1)2^{1-m})) + i\sin(x\sin(\pi(2p+1)2^{1-m})) \\
&\quad + \cos(x\sin(\pi(2p+1)2^{1-m})) - i\sin(x\sin(\pi(2p+1)2^{1-m}))) \\
&\quad = e^{x\cos(\pi(2p+1)2^{1-m})}\left(2\cos(x\sin(\pi(2p+1)2^{1-m}))\right) \\
&\quad = 2e^{x\cos(\pi(2p+1)2^{1-m})}\cos(x\sin(\pi(2p+1)2^{1-m}))
\end{aligned}
$$

Our summation is therefore

$$
\begin{aligned}
E_{m,0}(x) &= \frac{1}{\lceil 2^{m-1}\rceil}\sum_{p=0}^{\lceil 2^{m-2}\rceil-1} 2e^{x\cos(\pi(2p+1)2^{1-m})}\cos(x\sin(\pi(2p+1)2^{1-m})) \\
&= \frac{1}{\lceil 2^{m-2}\rceil}\sum_{p=0}^{\lceil 2^{m-2}\rceil-1} e^{x\cos(\pi(2p+1)2^{1-m})}\cos(x\sin(\pi(2p+1)2^{1-m}))
\end{aligned}
$$

The next step is to collapse the $e^{x\cos(\pi(2p+1)2^{1-m})}$ factors to produce $\cosh(x)$. We proceed as above, splitting the series in two, reparameterizing, reversing direction and grouping terms.

$$\begin{aligned}
&E_{m,0}(x)\\
&\quad= \frac{1}{\lceil 2^{m-2}\rceil} \sum_{p=0}^{\lceil 2^{m-2}\rceil-1} e^{x\cos(\pi(2p+1)2^{1-m})} \cos(x\sin(\pi(2p+1)2^{1-m}))\\
&\quad= \frac{1}{\lceil 2^{m-2}\rceil}\Big(\sum_{p=0}^{\lceil 2^{m-3}\rceil-1} e^{x\cos(\pi(2p+1)2^{1-m})} \cos(x\sin(\pi(2p+1)2^{1-m}))\\
&\quad+ \sum_{p=\lceil 2^{m-3}\rceil}^{\lceil 2^{m-2}\rceil-1} e^{x\cos(\pi(2p+1)2^{1-m})} \cos(x\sin(\pi(2p+1)2^{1-m}))\Big)
\end{aligned}$$

Focusing on the second summation for the moment, we find

$$\begin{aligned}
&\sum_{p=\lceil 2^{m-3}\rceil}^{\lceil 2^{m-2}\rceil-1} e^{x\cos(\pi(2p+1)2^{1-m})} \cos(x\sin(\pi(2p+1)2^{1-m}))\\
&\quad= \sum_{p=0}^{\lceil 2^{m-3}\rceil-1} e^{x\cos(\pi(2(p+2^{m-3})+1)2^{1-m})} \cos(x\sin(\pi(2(p+2^{m-3})+1)2^{1-m}))\\
&\quad= \sum_{p=0}^{\lceil 2^{m-3}\rceil-1} e^{x\cos(\pi(2p+1)2^{1-m}+\pi/2)} \cos(x\sin(\pi(2p+1)2^{1-m}+\pi/2))
\end{aligned}$$

We now reverse the order of summation by substituting $\lceil 2^{m-3}\rceil - 1 - p$ for p.

$$\begin{aligned}
&\sum_{p=0}^{\lceil 2^{m-3}\rceil-1} e^{x\cos(\pi(2p+1)2^{1-m}+\pi/2)} \cos(x\sin(\pi(2p+1)2^{1-m}+\pi/2))\\
&\quad= \sum_{p=0}^{\lceil 2^{m-3}\rceil-1} e^{x\cos(\pi(2(\lceil 2^{m-3}\rceil-1-p)+1)2^{1-m}+\pi/2)}\\
&\quad\cdot \cos(x\sin(\pi(2(\lceil 2^{m-3}\rceil-1-p)+1)2^{1-m}+\pi/2))\\
&\quad= \sum_{p=0}^{\lceil 2^{m-3}\rceil-1} e^{x\cos(\pi-\pi(2p+1)2^{1-m})} \cos(x\sin(\pi-\pi(2p+1)2^{1-m}))
\end{aligned}$$

Two useful identities are

$$\cos(\pi - k) = -\cos(k)$$

$$\sin(\pi - k) = -\sin(k)$$

Using these identities, the summation reduces further to

$$\begin{aligned}
&\sum_{p=0}^{\lceil 2^{m-3}\rceil - 1} e^{x\cos(\pi - \pi(2p+1)2^{1-m})} \cos(x \sin(\pi - \pi(2p+1)2^{1-m})) \\
&\quad = \sum_{p=0}^{\lceil 2^{m-3}\rceil - 1} e^{-x\cos(\pi(2p+1)2^{1-m})} \cos(-x \sin(\pi(2p+1)2^{1-m})) \\
&\quad = \sum_{p=0}^{\lceil 2^{m-3}\rceil - 1} e^{-x\cos(\pi(2p+1)2^{1-m})} \cos(x \sin(\pi(2p+1)2^{1-m}))
\end{aligned}$$

Plugging the above back into Equation (C.16), we find

$$\begin{aligned}
&E_{m,0}(x) \\
&\quad = \frac{1}{\lceil 2^{m-2}\rceil} \sum_{p=0}^{\lceil 2^{m-2}\rceil - 1} e^{x\cos(\pi(2p+1)2^{1-m})} \cos(x \sin(\pi(2p+1)2^{1-m})) \\
&\quad = \frac{1}{\lceil 2^{m-2}\rceil} \sum_{p=0}^{\lceil 2^{m-3}\rceil - 1} \left(e^{x\cos(\pi(2p+1)2^{1-m})} + e^{-x\cos(\pi(2p+1)2^{1-m})}\right) \cos(x \sin(\pi(2p+1)2^{1-m}))
\end{aligned}$$

Collapsing the exponential sum, we have

$$\begin{aligned}
E_{m,0}(x) &= \frac{1}{\lceil 2^{m-3}\rceil} \sum_{p=0}^{\lceil 2^{m-3}\rceil - 1} \cosh(x\cos(\pi(2p+1)2^{1-m})) \cos(x \sin(\pi(2p+1)2^{1-m})) \\
&= \frac{1}{\lceil 2^{m-3}\rceil} \frac{1}{\lceil 2^{m-3}\rceil} \cosh(xc_p) \cos(xs_p)
\end{aligned}$$

As advertised. Equation (C.16) holds for $m \geq 2$; for $m \leq 0$ we of course have simply $E_{m,0} = e^x$, and for $m = 1$ we have $E_{1,0} = e^{-x}$.

We now know how to express $E_{m,0}(x)$ in terms of the $\cosh(x)$ and $\cos(x)$ functions. Given this, all $E_{m,n}(x)$ can be found by repeated differentiation or integration.

In the next section, we will see another means to compute $E_{mn}(x)$.

C.2 $E_{mn}(x)$

By definition,

$$
\begin{aligned}
&E_{m,0}(x)\\
&\quad\equiv \frac{1}{\lceil 2^{m-1}\rceil}\sum_{p=0}^{\lceil 2^{m-1}\rceil-1} e^{xi^{(2p+1)2^{m-2}}}\\
&\quad= \frac{1}{\lceil 2^{m-1}\rceil}\sum_{p=0}^{\lceil 2^{m-1}\rceil-1} e^{x\cos((2p+1)\pi 2^{m-1})}e^{ix\sin((2p+1)\pi 2^{m-1})}\\
&\quad= \frac{1}{\lceil 2^{m-1}\rceil}\sum_{p=0}^{\lceil 2^{m-1}\rceil-1} e^{c_p x}e^{is_p x}\\
&\quad= \frac{1}{\lceil 2^{m-1}\rceil}\sum_{p=0}^{\lceil 2^{m-1}\rceil-1} \left(\cosh(c_p x)+\sinh(c_p x)\right)\left(\cos(s_p x)+i\sin(s_p x)\right)\\
&\quad= \frac{1}{\lceil 2^{m-1}\rceil}\sum_{p=0}^{\lceil 2^{m-1}\rceil-1}\\
&\quad\cosh(c_p x)\cos(s_p x)+\sinh(c_p x)\cos(s_p x)\\
&\quad+i\cosh(c_p x)\sin(s_p x)+i\sinh(c_p x)\sin(s_p x)
\end{aligned}
$$

Since each integration brings down a factor of $i^{-(2p+1)2^{m-2}}$ for the p^{th} term, we have

$$
\begin{aligned}
&E_{mn}(x)\\
&\quad= \frac{1}{\lceil 2^{m-1}\rceil}\sum_{p=0}^{\lceil 2^{m-1}\rceil-1} i^{-n(2p+1)2^{m-2}}\\
&\quad\cdot(\cosh(c_p x)\cos(s_p x)+\sinh(c_p x)\cos(s_p x)\\
&\quad+i\cosh(c_p x)\sin(s_p x)+i\sinh(c_p x)\sin(s_p x))
\end{aligned}
$$

Appendix D

The Orthogonality of $\psi_{mn}(x)$ Coefficients

This appendix proves the orthogonality of the vectors defined by Equation (6.2)

$$\vec{v}_{m,n,L}(t) = \begin{cases} (-1)^{t \div 2^{m-1}} & \text{if } n = t \mod 2^{m-1} \text{ and } 0 \leq t < L; \\ 0 & \text{Otherwise.} \end{cases} \tag{D.1}$$

for

$$M \geq 0 \tag{D.2}$$
$$L = 2^M \tag{D.3}$$
$$m = 0, 1, \ldots, M \tag{D.4}$$
$$n = 0, 1, \ldots, \lceil 2^{m-1} \rceil - 1 \tag{D.5}$$

That is,

$$\vec{v}_{m_0,n_0,L} \circ \vec{v}_{m_1,n_1,L} = 0 \quad \text{Unless } m_0 = m_1 \text{ and } n_0 = n_1 \tag{D.6}$$

To prove this, we need only consider vector components t in which both $\vec{v_0}$ and $\vec{v_1}$ both have non-zero values. This occurs when t is such that

$$0 \le t < L \tag{D.7}$$
$$n_0 = t \mod 2^{m_0} \tag{D.8}$$
$$n_1 = t \mod 2^{m_1} \tag{D.9}$$

The first observation is that any two vectors in the same level (*i.e.*, same m-value) are orthogonal. If $m_0 = m_1$ then $n_0 = n_1$, which says that no two distinct vectors at the same level share any non-zero entries. This is apparent from the table provided in Chapter 5.

For $m_0 \neq m_1$, assume without loss of generality that $m_1 > m_0$. Equation (6.2) allows us to break a vector into exactly $L/2^m$ intervals, each with the same pattern of zero and non-zero terms, with only a sign alternation between successive intervals for $m > 0$. The definition of L ensures that the number of intervals so defined will be even and ≥ 2.

We note that since $m_1 > m_0$ by assumption, $\vec{v_0}$ has more intervals than $\vec{v_1}$, by a factor of $2^{m_1}/2^{m_0}$.

We have the option of expressing $\vec{v_0}$ as the sum of $2^{m_1}/2^{m_0}$ vectors, constructed as follows

$$\vec{v_0}(t) = \sum_{q=0}^{2^{m_1-m_0}-1} \vec{w}_{m_0,n_0,q,L}(t)$$

where

$$\vec{w}_{m_0,n_0,q,L}(t) = \begin{cases} \vec{v}_{m_0,n_0,L}(t) & \text{iff } t \mod 2^{m_1} = q; \\ 0 & \text{Otherwise.} \end{cases} \tag{D.10}$$

The above has the effect of breaking $\vec{v_0}$ into a sum of offset vectors, each having the same interval as $\vec{v_1}$. $\vec{v_0} \circ \vec{v_1}$ can now be expressed as the sum of the inner products of $\vec{v_1}$ with each of the $\vec{w}_{m_0,n_0,q,L}$. These inner products can be considered separately.

That is,

$$\vec{v}_1 \circ \vec{v}_0 = \vec{v}_1 \circ \sum_{q=0}^{2^{m_1-m_0}-1} \vec{w}_{m_0,n_0,q,L} \tag{D.11}$$

Since each of the $\vec{w}_{m_0,n_0,q,L}$ has the same interval length as $\vec{v_1}$, $\vec{v_1} \circ \vec{w}_{m_0,n_0,q,L}$ will have no non-zero terms except when both $n_0 = n_1 \mod 2^{m_0}$ (*i.e.,* there are overlapping non-zero terms between $\vec{v}_0$ and $\vec{v}_1$) and $n_1 = n_0 + q$ (*i.e.,* the right decomposition of $\vec{v}_0$).

Thus, proving orthogonality reduces to proving $\vec{v_1} \circ \vec{w}_{m_0,n_0,q,L}$ for $n_0 = n_1 \mod 2^{m_0}$ and $n_1 = n_0 + q$, since all other inner products have no common non-zero terms.

Note that $\vec{v_1}$ has an even number of non-zero terms, half of these $+1$ and half -1.

$\vec{w}_{m_0,n_0,q,L}$ has exactly the same positions holding non-zero values as $\vec{v_1}$, since both have the same first position for a non-zero term and the same step size. However, while $\vec{v_1}$ has an equal number of positive and negative entries, all entries in $\vec{w}_{m_0,n_0,q,L}$ are of the same sign. This is because the terms of $\vec{w}_{m_0,n_0,q,L}$ are pulled from positions of $\vec{v_0}$ that are an even number of intervals apart.

Therefore, $\vec{v_1} \circ \vec{w}_{m_0,n_0,q,L} = 0$, which completes the proof that the vectors defined by Equation (6.2) are orthogonal.

Appendix E

Derivation of the Broom Theorems

This appendix proves the theorems asserted in Chapter 7, having to do with translating between levels of Cairns space.

E.1 Sweeping Up

Let $m_v > m_u \geq 0$ and define $c \equiv 2^{m_v - 1}/\lceil 2^{m_u - 1} \rceil$.

By definition,

$$\psi_{m_u,n}(x) = \sum_{t=0}^{\infty} (-1)^{t\lceil 2^{1-m_u} \rceil} \frac{x^{t\lceil 2^{m_u-1} \rceil + n}}{(t\lceil 2^{m_u-1} \rceil + n)!} \tag{E.1}$$

Let us perform a change of variables.

$$\psi_{m_u,n}(x) = \sum_{s=0}^{\infty} \sum_{t=0}^{c-1} (-1)^{t\lceil 2^{1-m_u} \rceil} \frac{x^{s2^{m_v-1} + t\lceil 2^{m_u-1} \rceil + n}}{(s2^{m_v-1} + t\lceil 2^{m_u-1} \rceil + n)!} \tag{E.2}$$

This creates "big steps" of size 2^{m_v-1}, and covers the terms stepped over with "small steps" of size $\lceil 2^{m_u-1} \rceil$.

As a short-hand, define

$$q \equiv s2^{m_v-1} + t\lceil 2^{m_u-1} \rceil + n$$

Then, simply re-expressing the above, we have

$$\psi_{m_u,n}(x) = \sum_{s=0}^{\infty} \sum_{t=0}^{c-1} (-1)^{t\lceil 2^{1-m_u} \rceil} \frac{x^q}{q!} \tag{E.3}$$

Now, switch the order of summation:

$$\psi_{m_u,n}(x) = \sum_{t=0}^{c-1} (-1)^{t\lceil 2^{1-m_u} \rceil} \sum_{s=0}^{\infty} \frac{x^q}{q!} \tag{E.4}$$

The summation

$$\sum_{s=0}^{\infty} \frac{x^q}{q!}$$

could be expressed at level m_v if it had a sign alternation of $(-1)^s$. We can introduce a sign alternation by introducing a factor inside x^q that cancels a sign alternation outside x^q. This can be done as follows.

$$x^q = \left(i^{\left(2^{2-m_v}\right)} x\right)^q i^{-q2^{2-m_v}} \tag{E.5}$$

Expanding the definition of q and rearranging, we have

$$i^{-q2^{2-m_v}} = \left(i^{-2s}\right) \left(i^{-(t\lceil 2^{m_u-1} \rceil + n)2^{2-m_v}}\right) \tag{E.6}$$

$$= -1^s \left(i^{-(t\lceil 2^{m_u-1} \rceil + n)2^{2-m_v}}\right) \tag{E.7}$$

This gives us what we need. The first factor of the above gives us a sign alternation for Equation (E.4). Since the second factor has no s-dependency it can be moved to the outer (first) summation. Therefore we have

$$\psi_{m_u,n}(x) = \sum_{t=0}^{c-1} (-1)^{t\lceil 2^{1-m_u} \rceil} \left(i^{-(t\lceil 2^{m_u-1} \rceil + n)2^{2-m_v}}\right) \sum_{s=0}^{\infty} (-1)^s \frac{(i^{(2^{2-m_v})} x)^q}{q!}$$

$$(E.8)$$

Here $t\lceil 2^{m_u-1}\rceil + n$ plays the role of n at level m_v, varying across t. Putting it together, we have

$$\psi_{m_u,n}(x) = \sum_{t=0}^{c-1} i^{(-t\lceil 2^{m_u-1}\rceil+n)/\lceil 2^{m_v-2}\rceil} (-1)^{t\lceil 2^{m_u-1}\rceil} \cdot \psi_{(m_v,t\lceil 2^{m_u-1}\rceil+n)}(xi^{(2^{2-m_v})}) \tag{E.9}$$

which allows us to express $\psi_{m_u,n}(x)$ in terms of functions at level m_v, for $m_v > m_u \geq 0$.

E.2 Sweeping Sideways

This section proves Equation (7.3), for expressing a function at level m in terms of other functions at level m.

Let us examine a special case at $m = 3$ to understand the idea. From the definition of a Taylor series,

$$\psi_{3,0}(x) \quad \text{(E.10)}$$
$$= \psi_{3,0}(a) \quad \text{(E.11)}$$
$$+(x-a)\psi\prime_{3,0}(a) \quad \text{(E.12)}$$
$$+\frac{(x-a)^2}{2!}\psi\prime\prime_{3,0}(a) \quad \text{(E.13)}$$
$$+\frac{(x-a)^3}{3!}\psi\prime\prime\prime_{3,0}(a) \quad \text{(E.14)}$$
$$+\dots \quad \text{(E.15)}$$
$$= \psi_{3,0}(a) \quad \text{(E.16)}$$
$$+(x-a)(-\psi_{3,3}(a)) \quad \text{(E.17)}$$
$$+\frac{(x-a)^2}{2!}(-\psi_{3,2}(a)) \quad \text{(E.18)}$$
$$+\frac{(x-a)^3}{3!}(-\psi_{3,1}(a)) \quad \text{(E.19)}$$
$$+\dots \quad \text{(E.20)}$$
$$= \psi_{3,0}(a)\psi_{3,0}(x-a) \quad \text{(E.21)}$$
$$-\psi_{3,3}(a)\psi_{3,1}(x-a) \quad \text{(E.22)}$$
$$-\psi_{3,2}(a)\psi_{3,2}(x-a) \quad \text{(E.23)}$$
$$-\psi_{3,1}(a)\psi_{3,3}(x-a) \quad \text{(E.24)}$$

Notice that in the special case $a = 0$ we have $\psi_{3,0}(a) = 1$, $\psi_{3,1}(a) = \psi_{3,2}(a) = \psi_{3,3}(a) = 0$ and the above reduces to a simple identity.

We now apply the same approach more generally.

$$\psi_{m,n}(x) = \sum_{t=0}^{\infty} \frac{(x-a)^t}{t!}\psi^{(t)}_{m,n}(a) \quad \text{(E.25)}$$
$$= \sum_{p=0}^{\lceil 2^{m-1}\rceil -1} \sum_{t=0}^{\infty} \frac{(x-a)^{t\lceil 2^{m-1}\rceil + p}}{(t\lceil 2^{m-1}\rceil + p)!}\psi^{(t\lceil 2^{m-1}\rceil + p)}_{m,n}(a) \quad \text{(E.26)}$$

Next, take advantage of the properties of derivatives of $\psi_{mn}(x)$ (see Chapter 4), specifically that $\psi\prime_{m,n}(x) = \psi_{m,n-1}(x)$ if $n \geq 1$, and $\psi\prime_{m,0}(x) = -\psi_{m,\lceil 2^{m-1}\rceil -1}(x)$. Consequently, in the first cycle of derivatives in Equation (E.25) the sign will be positive for the first

n differentiations, and negative thereafter. This pattern will reverse in the next cycle, and alternate thereafter.

We also know that there are exacly $\lceil 2^{m-1} \rceil - 1$ functions at level m, meaning that after $\lceil 2^{m-1} \rceil - 1$ differentiations we return to the same function, with a sign change.

Combining this information, we have

$$\psi_{m,n}(x) = \sum_{p=0}^{\lceil 2^{m-1} \rceil - 1} \sum_{t=0}^{\infty} \frac{(x-a)^{t\lceil 2^{m-1} \rceil + p}}{(t\lceil 2^{m-1} \rceil + p)!} (-1)^t \,\text{whole}(n-p) \psi_{m,(n-p) mod \lceil 2^{m-1} \rceil - 1}(a)$$

Where, as in Chapter 7 we define whole$(n-p) = 1$ for $(n-p) \geq 0$ and -1 otherwise.

Finally, we note that by the definition of $\psi_{mn}(x)$ (see Chapter 4) we can write

$$\psi_{m,n}(x) = \sum_{p=0}^{\lceil 2^{m-1} \rceil - 1} \text{whole}(n-p) \psi_{m,p}(x-a) \psi_{m,(n-p) mod \lceil 2^{m-1} \rceil - 1}(a)$$

Appendix F

Derivation of the Inner Product

This appendix derives Equation (8.24) for the inner product at level m.

$$
\begin{aligned}
& E_{m,n}(x) \circ E_{m,n}(x+\alpha) && \text{(F.1)} \\
& \quad = \sum_{n=0}^{\lceil 2^{m-1} \rceil - 1} E_{m,n}(x) E_{m,n}(x+\alpha) && \text{(F.2)} \\
& \quad = \frac{1}{\lceil 2^{m-1} \rceil} \sum_{p=0}^{\lceil 2^{m-1} \rceil - 1} e^{2x \cos(\pi(2p+1)2^{1-m})} e^{\alpha i^{-(2p+1)2^{2-m}}} && \text{(F.3)}
\end{aligned}
$$

for the inner product at level m.

By definition, the left side of the above is equivalent to

$$E_{m,n}(x) \circ E_{m,n}(x+\alpha)$$

$$= \sum_{n=0}^{\lceil 2^{m-1}\rceil - 1} E_{m,n}(x) E_{m,n}(x+\alpha)$$

$$= \sum_{n=0}^{\lceil 2^{m-1}\rceil - 1} E_{m,n}(x) E_{m,n}(x+\alpha)$$

$$= \sum_{n=0}^{\lceil 2^{m-1}\rceil - 1}$$

$$\left(\frac{1}{\lceil 2^{m-1} \rceil} \sum_{p_a=0}^{\lceil 2^{m-1}\rceil - 1} i^{-n(2p_a+1)2^{2-m}} e^{xi^{(2p_a+1)2^{2-m}}} \right)$$

$$\left(\frac{1}{\lceil 2^{m-1} \rceil} \sum_{p_b=0}^{\lceil 2^{m-1}\rceil - 1} i^{-n(2p_b+1)2^{2-m}} e^{(x+\alpha)i^{(2p_b+1)2^{2-m}}} \right)$$

This summation is fairly horrendous. Cats have choked on less. The first order of business is to reverse the order of summations, to temporarily "freeze" the exponent. Otherwise, one is in for a very long night.

Collecting the summations and the coefficients, and bringing the "n" summation to the inside, we obtain

$$\sum_{p_a=0}^{\lceil 2^{m-1}\rceil - 1} \sum_{p_b=0}^{\lceil 2^{m-1}\rceil - 1} \sum_{n=0}^{\lceil 2^{m-1}\rceil - 1} \left(\frac{1}{\lceil 2^{m-1} \rceil} \right)^2 i^{-n(2p_a+1)2^{2-m}} i^{-n(2p_b+1)2^{2-m}}$$

$$e^{xi^{(2p_a+1)2^{2-m}}} e^{(x+\alpha)i^{(2p_b+1)2^{2-m}}}$$

The coefficients depend on n, but the exponents do not. So we can examine the inner summation considering the exponential factor to be constant.

As is often true, it is useful to look at $m = 3$ explicitly to understand the general pattern. If we look at the factor

$$i^{-n(2p+1)/2} \tag{F.4}$$

for $m = 3$, we can build the following table showing how it varies with n and p.

	p=0	p=1	p=2	p=3
n=0	1	1	1	1
$n = 1$	$i^{-1/2}$	$i^{-3/2}$	$-i^{-1/2}$	$-i^{-3/2}$
$n = 2$	i^{-1}	$-i^{-1}$	i^{-1}	$-i^{-1}$
$n = 3$	$i^{-3/2}$	$i^{-1/2}$	$-i^{-3/2}$	$-i^{-1/2}$

Since p_a and p_b are fixed for the inner summation, summing over n amounts to multiplying some combination of two columns from the above table together and adding down the rows of those two columns.

It is notable that this procedure will yield zero for all combinations of two columns except $(p = 0, p = 3)$ and $(p = 1, p = 2)$. These happen to be the only combinations for which $p_a + p_b = \lceil 2^{m-1} \rceil - 1$.

The same pattern holds in the simpler case of $m = 2$, where the corresponding matrix is

	$p = 0$	$p = 1$
$n = 0$	1	1
$n = 1$	i^{-1}	i^{-3}

Does this pattern generalize for $m \geq 4$?

Let us examine the summation of the coefficients by themselves.

$$\sum_{n=0}^{\lceil 2^{m-1} \rceil -1} i^{-n(2p_a+1)2^{2-m}} i^{-n(2p_b+1)2^{2-m}} = \tag{F.5}$$

$$\sum_{n=0}^{\lceil 2^{m-1} \rceil -1} i^{-2n(p_a+p_b+1)2^{2-m}} \tag{F.6}$$

For $m \geq 2$ we can split the summation in two:

$$\sum_{n=0}^{\lceil 2^{m-1}\rceil -1} i^{-2n(p_a+p_b+1)2^{2-m}} \tag{F.7}$$

$$= \sum_{n=0}^{\lceil 2^{m-2}\rceil -1} i^{-2n(p_a+p_b+1)2^{2-m}} + \sum_{n=\lceil 2^{m-2}\rceil}^{\lceil 2^{m-1}\rceil -1} i^{-2n(p_a+p_b+1)2^{2-m}} \tag{F.8}$$

$$= \sum_{n=0}^{\lceil 2^{m-2}\rceil -1} \left(i^{-2n(p_a+p_b+1)2^{2-m}}\right)\left(1 + i^{-2(p_a+p_b+1)}\right) \tag{F.9}$$

The above summation will be zero whenever $p_a + p_b + 1$ is odd, since the second factor will be zero. If $p_a + p_b + 1$ is even then the second factor reduces to 2.

If we assume $p_a + p_b + 1$ is even then for $m \geq 3$ we can again split the summation in two, precisely as above. We obtain

$$\sum_{n=0}^{\lceil 2^{m-3}\rceil -1} 2\left(i^{-2n(p_a+p_b+1)2^{2-m}}\right)\left(1 + i^{-2(p_a+p_b+1)}\right)$$

We saw at the previous step that $p_a + p_b + 1$ must be even for non-zero cases. This step tells us that only even terms which are divisible by four produce a non-zero sum.

We can repeat this process recursively as many times as 2^{m-1} can be divided into two equal parts, with at least one member each; that is, $m - 1$ times. After $m - 1$ steps, we know that $p_a + p_b + 1$ must be a multiple of 2^{m-1}.

This provides the generalization of the $m = 2$ and $m = 3$ results we saw from the above tables. The summation is now less horrendous. From here, it requires only a cigar and a shot of Pusser's to complete the derivation of the inner product formula.

Because we know that only cases where $p_a + p_b + 1 = 2^{m-1}$ will have a non-zero sum, we can remove all other cases from the summation without changing the result. Let us therefore define

$$p_b = 2^{m-1} - p_a - 1 \tag{F.10}$$

Which gives us

$$
\begin{aligned}
(2p_b+1)2^{2-m} &= (2(2^{m-1}-p_a-1)+1)2^{2-m} && \text{(F.11)}\\
&= (2^m-2p_a-1)2^{2-m} && \text{(F.12)}\\
&= 4-(2p_a+1)2^{2-m} && \text{(F.13)}
\end{aligned}
$$

So

$$
\begin{aligned}
i^{(2p_b+1)2^{2-m}} &= i^{4-(2p_a+1)2^{2-m}} && \text{(F.14)}\\
&= i^{-(2p_a+1)2^{2-m}} && \text{(F.15)}
\end{aligned}
$$

To summarize, we now have the following simplifications available

1. The summation over n, and the imaginary coefficients, can be replaced by the constraint $p_a + p_b + 1 = 2^{m-1}$ and a factor of 2^{m-1}.
2. Since there is only one allowable value of p_b for a given value of p_a, we can eliminate p_b from the exponent and drop the summation over p_b.

This gives us

$$
\begin{aligned}
&\sum_{n=0}^{\lceil 2^{m-1}\rceil-1} E_{m,n}(x)E_{m,n}(x+\alpha)\\
&= \sum_{p_a=0}^{\lceil 2^{m-1}\rceil-1}\sum_{p_b=0}^{\lceil 2^{m-1}\rceil-1}\sum_{n=0}^{\lceil 2^{m-1}\rceil-1}\\
&\left(\frac{1}{\lceil 2^{m-1}\rceil}\right)^2 i^{-n(2p_a+1)2^{2-m}} i^{-n(2p_b+1)2^{2-m}}\\
&\lceil 2^{m-1}\rceil e^{xi^{(2p_a+1)2^{2-m}}} e^{(x+\alpha)i^{(2p_b+1)2^{2-m}}}\\
&= \sum_{p_a=0}^{\lceil 2^{m-1}\rceil-1}\left(\frac{1}{\lceil 2^{m-1}\rceil}\right)^2 \lceil 2^{m-1}\rceil e^{xi^{(2p_a+1)2^{2-m}}} e^{(x+\alpha)i^{-(2p_a+1)2^{2-m}}}
\end{aligned}
$$

Cancelling and regrouping, we have

$$\sum_{p_a=0}^{\lceil 2^{m-1}\rceil-1} \frac{1}{\lceil 2^{m-1}\rceil} e^{x\left(i^{(2p_a+1)2^{2-m}}+i^{-(2p_a+1)2^{2-m}}\right)} e^{\alpha i^{-(2p_a+1)2^{2-m}}}$$

Finally, we can rewrite the imaginary sum as follows. In general,

$$\begin{aligned} i^a + i^{-a} &= e^{i\pi a/2} + e^{-i\pi a/2} \\ &= (\cos(\pi a/2) + i\sin(\pi a/2))\,(\cos(\pi a/2) - i\sin(\pi a/2)) \\ &= 2cos(\pi a/2) \end{aligned}$$

Using this identity for $a = (2p+1)2^{2-m}$ and substituting p for p_a, we have

$$\frac{1}{\lceil 2^{m-1}\rceil} \sum_{p=0}^{\lceil 2^{m-1}\rceil-1} e^{2x\cos(\pi(2p+1)2^{1-m})} e^{\alpha i^{-(2p+1)2^{2-m}}}$$

as claimed.

Appendix G

The Cairns Functions For $m \leq 4$

This appendix details $\psi_{mn}(x)$ and $E_{mn}(x)$ for the levels $m \leq 4$. Recall that

$$\psi_{mn}(x) = \sum_{t=0}^{\infty} (-1)^{t\lceil 2^{1-m}\rceil} \frac{x^{t\lceil 2^{m-1}\rceil + n}}{(t\lceil 2^{m-1}\rceil + n)!} \quad (G.1)$$

$$E_{m,0}(x) = \frac{1}{\lceil 2^{m-1}\rceil} \sum_{p=0}^{\lceil 2^{m-1}\rceil - 1} e^{xi^{(2p+1)2^{2-m}}} \quad (G.2)$$

$$(G.3)$$

and

$$e^{xi^{\left(2^{2-m}\right)}} = \sum_{n=0}^{\lceil 2^{m-1}\rceil - 1} i^{n2^{2-m}} \left(\psi_{mn}(x) = E_{mn}(x)\right) \quad (G.4)$$

G.1 $m \leq 0$

For $m = 0$,

$$\psi_{0,0}(x) = \sum_{t=0}^{\infty} \frac{x^t}{t!}$$

and

$$E_0, 0(x) = e^x$$

G.2 $m = 1$

For $m = 1$,

$$\psi_{1,0}(x) = \sum_{t=0}^{\infty} -1^t \frac{x^t}{t!}$$

and

$$E_{1,0}(x) = e^{-x}$$

G.3 $m = 2$

For $m = 2$,

$$\psi_{2,0}(x) = \sum_{t=0}^{\infty} -1^t \frac{x^{2t}}{(2t)!}$$

$$\psi_{2,1}(x) = \sum_{t=0}^{\infty} -1^t \frac{x^{2t+1}}{(2t+1)!}$$

and

$$E_{2,0}(x) = \cos(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$$

$$E_{2,1}(x) = \sin(x) = \frac{1}{2i}\left(e^x - e^{-x}\right)$$

G.4 $m = 3$

For $m = 3$,

$$\psi_{3,0}(x) = \sum_{t=0}^{\infty} -1^t \frac{x^{4t}}{(4t)!}$$

$$\psi_{3,1}(x) = \sum_{t=0}^{\infty} -1^t \frac{x^{4t+1}}{(4t+1)!}$$

$$\psi_{3,2}(x) = \sum_{t=0}^{\infty} -1^t \frac{x^{4t+2}}{(4t+2)!}$$

$$\psi_{3,3}(x) = \sum_{t=0}^{\infty} -1^t \frac{x^{4t+3}}{(4t+3)!}$$

The $E_{mn}(x)$values can be found from the below table, generated by integrating $4E_{3,0} = e^{xi^{1/2}} + e^{xi^{3/2}} + e^{xi^{5/2}} + e^{xi^{7/2}}$

	$e^{xi^{1/2}}$	$e^{xi^{3/2}}$	$e^{xi^{5/2}}$	$e^{xi^{7/2}}$
$4E_{3,0}(x)$	1	1	1	1
$4E_{3,1}(x)$	$i^{-1/2}$	$i^{-3/2}$	$-i^{-1/2}$	$-i^{-3/2}$
$4E_{3,2}(x)$	i^{-1}	$-i^{-1}$	i^{-1}	$-i^{-1}$
$4E_{3,3}(x)$	$i^{-3/2}$	$i^{-1/2}$	$-i^{-3/2}$	$-i^{-1/2}$

Alternatively, by differentiation of Equation (5.9) we have

$$\begin{aligned}
E_{3,0} &= \cosh(\frac{x}{\sqrt{2}})\cos(\frac{x}{\sqrt{2}}) \\
E_{3,1} &= \frac{1}{\sqrt{2}}\left(\cosh(\frac{x}{\sqrt{2}})\sin(\frac{x}{\sqrt{2}}) + \sinh(\frac{x}{\sqrt{2}})\cos(\frac{x}{\sqrt{2}})\right) \\
E_{3,2} &= \sinh(\frac{x}{\sqrt{2}})\sin(\frac{x}{\sqrt{2}}) \\
E_{3,3} &= \frac{1}{\sqrt{2}}\left(\cosh(\frac{x}{\sqrt{2}})\sin(\frac{x}{\sqrt{2}}) - \sinh(\frac{x}{\sqrt{2}})\cos(\frac{x}{\sqrt{2}})\right)
\end{aligned}$$

G.5 $m = 4$

For $m = 4$,

$$\psi_{4,0}(x) = \sum_{t=0}^{\infty} -1^t \frac{x^{8t}}{(8t)!}$$

$$\psi_{4,1}(x) = \sum_{t=0}^{\infty} -1^t \frac{x^{8t+1}}{(8t+1)!}$$

$$\psi_{4,2}(x) = \sum_{t=0}^{\infty} -1^t \frac{x^{8t+2}}{(8t+2)!}$$

$$\psi_{4,3}(x) = \sum_{t=0}^{\infty} -1^t \frac{x^{8t+3}}{(8t+3)!}$$

$$\psi_{4,4}(x) = \sum_{t=0}^{\infty} -1^t \frac{x^{8t+4}}{(8t+4)!}$$

$$\psi_{4,5}(x) = \sum_{t=0}^{\infty} -1^t \frac{x^{8t+5}}{(8t+5)!}$$

$$\psi_{4,6}(x) = \sum_{t=0}^{\infty} -1^t \frac{x^{8t+6}}{(8t+6)!}$$

$$\psi_{4,7}(x) = \sum_{t=0}^{\infty} -1^t \frac{x^{8t+7}}{(8t+7)!}$$

and for $E_{mn}(x)$we have

	$e^{xi^{1/4}}$	$e^{xi^{3/4}}$	$e^{xi^{5/4}}$	$e^{xi^{7/4}}$	$e^{xi^{9/4}}$	$e^{xi^{11/4}}$	$e^{xi^{13/4}}$	$e^{xi^{15/4}}$
$8E_{4,0}(x)$	1	1	1	1	1	1	1	1
$8E_{4,1}(x)$	$i^{-1/4}$	$i^{-3/4}$	$i^{-5/4}$	$i^{-7/4}$	$i^{-9/4}$	$i^{-11/4}$	$i^{-13/4}$	$i^{-15/4}$
$8E_{4,2}(x)$	$i^{-1/2}$	$i^{-3/2}$	$i^{-5/2}$	$i^{-7/2}$	$i^{-1/2}$	$i^{-3/2}$	$i^{-5/2}$	$i^{-7/2}$
$8E_{4,3}(x)$	$i^{-3/4}$	$i^{-9/4}$	$i^{-15/4}$	$i^{-5/4}$	$i^{-11/4}$	$i^{-1/4}$	$i^{-7/4}$	$i^{-13/4}$
$8E_{4,4}(x)$	i^{-1}	i^{-3}	i^{-1}	i^{-3}	i^{-1}	i^{-3}	i^{-1}	i^{-3}
$8E_{4,5}(x)$	$i^{-5/4}$	$i^{-15/4}$	$i^{-9/4}$	$i^{-3/4}$	$i^{-13/4}$	$i^{-7/4}$	$i^{-1/4}$	$i^{-11/4}$
$8E_{4,6}(x)$	$i^{-3/2}$	$i^{-1/2}$	$i^{-7/2}$	$i^{-5/2}$	$i^{-3/2}$	$i^{-1/2}$	$i^{-7/2}$	$i^{-5/2}$
$8E_{4,7}(x)$	$i^{-7/4}$	$i^{-5/4}$	$i^{-3/4}$	$i^{-1/4}$	$i^{-15/4}$	$i^{-13/4}$	$i^{-11/4}$	$i^{-9/4}$

Alternatively, we can differentiate Equation (5.9). For conciseness, we make use of the abbreviations

$$\begin{aligned}
\alpha &\equiv \cos(\pi/8) = \sin(3\pi/8) && \text{(G.5)}\\
\beta &\equiv \sin(\pi/8) = \cos(3\pi/8) && \text{(G.6)}\\
c &\equiv \cos && \text{(G.7)}\\
s &\equiv \sin && \text{(G.8)}\\
c_h &\equiv \cosh && \text{(G.9)}\\
s_h &\equiv \sinh && \text{(G.10)}
\end{aligned}$$

The following identities are also useful for simplifying expressions[1]

$$\begin{aligned}
\alpha^2 + \beta^2 &= 1 && \text{(G.11)}\\
\alpha^2 - \beta^2 &= \frac{1}{\sqrt{2}} && \text{(G.12)}\\
2\alpha\beta &= \frac{1}{\sqrt{2}} && \text{(G.13)}\\
(\alpha + \beta)/\sqrt{2} &= \alpha && \text{(G.14)}\\
(\alpha - \beta)/\sqrt{2} &= \beta && \text{(G.15)}
\end{aligned}$$

Using the above, we have

[1] These can be derived from the fact that $\alpha = (1 + \sqrt{2})/\left(\sqrt{2}\sqrt{\sqrt{2}(1+\sqrt{2})}\right)$ and $\beta = 1/\left(\sqrt{2}\sqrt{\sqrt{2}(1+\sqrt{2})}\right)$, as can seen by applying the double angle formula starting with $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$.

$$
\begin{aligned}
E_{4,0} &= c_h(\alpha x)c(\beta x) + c_h(\beta x)c(\alpha x)\\
E_{4,1} &= \beta c_h(\alpha x)s(\beta x) + \alpha s_h(\alpha x)c(\beta x) + \alpha c_h(\beta x)s(\alpha x) + \beta s_h(\beta x)c(\alpha x)\\
E_{4,2} &= \frac{1}{\sqrt{2}}\left(s_h(\alpha x)s(\beta x) + c_h(\alpha x)c(\beta x) + s_h(\beta x)s(\alpha x) - c_h(\beta x)c(\alpha x)\right)\\
E_{4,3} &= \alpha c_h(\alpha x)s(\beta x) + \beta s_h(\alpha x)c(\beta x) - \beta c_h(\beta x)s(\alpha x) - \alpha s_h(\beta x)c(\alpha x)\\
E_{4,4} &= s_h(\alpha x)s(\beta x) - s_h(\beta x)s(\alpha x)\\
E_{4,5} &= -\beta s_h(\alpha x)c(\beta x) + \alpha c_h(\alpha x)s(\beta x) + \alpha s_h(\beta x)c(\alpha x) - \beta c_h(\beta x)s(\alpha x)\\
E_{4,6} &= \frac{1}{\sqrt{2}}\left(-c_h(\alpha x)c(\beta x) + s_h(\alpha x)s(\beta x) + c_h(\beta x)c(\alpha x) + s_h(\beta x)s(\alpha x)\right)\\
E_{4,7} &= -\alpha s_h(\alpha x)c(\beta x) + \beta c_h(\alpha x)s(\beta x) - \beta s_h(\beta x)c(\alpha x) + \alpha c_h(\beta x)s(\alpha x)
\end{aligned}
$$

Notice that for even n, $E_{4,n}(x)$ consists of only cosh cos or sinh sin terms, whereas for odd n $E_{4,n}(x)$ consists of only cosh sin or sinh cos terms. This is consistent with the requirement that $E_{4,n}(x)$ must be symmetric for even n, and anti-symmetric for odd n.[2]

As an exercise, it is possible to use the broom theorems and a modest amount of trickery to show that $E_{4,0}(x) = 0$ is equivalent to

$$-1 = tanh(\alpha x/\sqrt{2})tanh(\beta x/\sqrt{2})tan(\alpha x/\sqrt{2})tan(\beta x/\sqrt{2})$$

An approximate solution is $x = 3.76$, as is evident from taking a two-term approximation of $\psi_{4,0}(x) = 0$.

[2]The question may arise as to whether it is possible to factor

$$E_{4,0} = c_h(\alpha x)c(\beta x) + c_h(\beta x)c(\alpha x)$$

into a single term, of the form $(a + b)(c + d)(e + f)$.

I believe the answer to be "no", on symmetry grounds. By inspection,

$$\psi_{4,0}(x) = 1 - \frac{x^8}{8!} + \frac{x^{16}}{16!} - \ldots$$

is invariant under the 7 distinct transforms $x \to i^{\frac{1}{2}}x$, $x \to ix$, $x \to i^{3/2}x$, $x \to i^2 x$, $x \to i^{5/2}x$, $x \to i^3 x$, $x \to i^{7/2}x$

Since $E_{4,0}(x) = \psi_{4,0}(x)$, it follows that $E_{4,0}(x)$ must have the same invariants. For $E_{3,0}(x)$, the corresponding 3 transforms essentially map a given element into each of the 3 other elements of $E_{3,0}(x) = \cosh(x)\cos(x) = (e^x + e^{-x})(e^{ix} + e^{-ix})/4$. But this would not work for $E_{4,0}(x) \stackrel{?}{=} (a+b)(c+d)(e+f)$, since there are only 5 other elements available for 7 transforms.